MSSL Advanced Theory Lectures

General Relativity - Problem Sheet 2

Question 1

By considering the action of the covariant derivative on a scalar, derive the definitions of the covariant derivative acting on both contravariant and covariant vectors. Further, from the metricity condition

$$\nabla_a \, g_{bc} = 0, \tag{1}$$

demonstrate the uniqueness of the affine connection (derive the Christoffel symbol).

Question 2

Consider the metric

$$ds^2 = -f(r)dt^2 + g(r)dr^2. \tag{2}$$

Derive the non-vanishing components of the Christoffel symbol. In two dimensions the Riemann curvature tensor has only one independent component, compute this component for the above metric.

Question 3

Recall the most general static, spherically symmetric metric of question 9, problem sheet 1

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{3}$$

Use your answers to question 9 of the previous problem sheet to derive the Ricci tensor components R_{ab} for this metric. From this derive the Ricci scalar R for the metric. Finally, compute the components of the Einstein tensor G_{ab}.

(The following results are useful: $R_{ab} = \Gamma^c{}_{ab,c} - \Gamma^c{}_{cb,a} + \Gamma^d{}_{ab}\Gamma^c{}_{dc} - \Gamma^d{}_{cb}\Gamma^c{}_{da}$, $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} \, R$, R_{ab} and G_{ab} are diagonal for the above metric.)

Question 4

For a perfect fluid in a spacetime governed by the metric in equation (3), the energy-momentum tensor takes the form

$$T_{ab} = (\rho + P)u_a u_b + P \, g_{ab}, \tag{4}$$

with 4-velocity $u_a = \begin{pmatrix} -e^{\nu(r)/2} \\ \vec{0} \end{pmatrix}^T$, where $\rho \equiv \rho(r)$, $P \equiv P(r)$. From the fluid conservation equation $\nabla_a T^{ab} = 0$ derive the equations of motion in this spacetime.

(Hint: only one component gives a non-trivial result)

Question 5

Recall the derivation of the Schwarzschild solution to the vacuum field equations in lecture 4. Now consider the vacuum field equations in the presence of the cosmological constant Λ:

$$G_{ab} + \Lambda g_{ab} = 0. \tag{5}$$

Demonstrate these equations are equivalent to

$$R_{ab} = \Lambda g_{ab}. \tag{6}$$

Now solve the vacuum field equations with cosmological constant to find

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{7}$$

Question 6

Consider the Faraday tensor

$$F_{ab} = A_{b,a} - A_{a,b}, \tag{8}$$

where A is a 4-potential. Demonstrate that F_{ab} is a covariant tensor of rank 2.
(Hint: consider the Faraday tensor in a curved spacetime)

Question 7

(i) For a vector u^a with $u_a u^a = -1$, show that the tensor P_{ab} defined as

$$P_{ab} = g_{ab} + u_a u_b, \tag{9}$$

is a projection tensor, namely that $P^b{}_a P^c{}_b = P^c{}_a$.

(ii) For any vector W^a, we can define the perpendicular component of that vector as

$$W^b_\perp = P^b{}_c W^c. \tag{10}$$

Show that $W^b_\perp$ is orthogonal to U^b.

Question 8

Consider the unit 2-sphere ($r = 1$), which is represented by the surface of a sphere, given by the following metric

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2. \tag{11}$$

Clearly $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, with $\theta = \pi/2$ corresponding to the equator and $\theta = 0,\ \pi$ corresponding to the North and South poles respectively.

(i) Compute the Christoffel symbols for this metric

(ii) Write down the equations of parallel transport on the surface of the sphere and solve them for both transport along curves $\theta = const.$ and $\phi = const.$, for parallel transport moving at constant angular velocity ω (i.e. angular velocities along the θ- and ϕ-constant curves are identical).

(iii) Consider a contravariant vector W^a parallelly transported along a closed path on the surface of the sphere. Start on the equator at the point $(\pi/2, 0)$ with $W^a_{initial} = (1, \overline{0})^T$. Now parallelly transport this vector along the equator to the point $(\pi/2, \varphi)$, then transport North along the $\phi = const.$ surface to the North pole, then finally move south along a $\phi = const.$ surface to the starting point $\pi/2, 0$, i.e. parallelly transport around a spherical triangle, as described in the lectures. Evaluate W^a_{final} and calculate the change in angle between the initial and final vectors.
(Hint: drawing a diagram will help visualise this)

Question 9

Two metrics g_{ab} and $\overline{g}_{ab}$ are said to be conformally related if $\overline{g}_{ab} = \Omega^2(x) g_{ab}$, where Ω is a scalar function. Demonstrate that the Christoffel symbols in the two metrics are related by

$$\overline{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + \Omega^{-1}(\delta^a_b\, \Omega_{,c} + \delta^a_c\, \Omega_{,b} - g^{ad} g_{bc}\, \Omega_{,d}). \tag{12}$$

Consider the conformally flat metric

$$\overline{g}_{ab} = e^{2\phi(x)} \eta_{ab}. \tag{13}$$

Compute the geodesic equations of motion for the above metric and find the corresponding Ricci tensor and Ricci scalar.

Question 10

As we have already discussed in the lectures, gravity is a manifestation of the curvature of spacetime by matter and energy. If we have a given mass distribution, which we prescribe with an appropriate stress-energy-momentum tensor in that spacetime, using the Einstein field equations the metric of the resulting spacetime may be derived. The field equations may be derived from an action, which has the key advantage that the field equations can be coupled with matter fields, as well as for General Relativity to be unified with other field theories (e.g. Electromagnetism). The most physical way to do this is with the Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g} \left(\frac{c^4}{16\pi G} R + \mathcal{L}_M \right). \tag{14}$$

The key to deriving the Einstein field equations is to treat the metric and the affine connection as independent degrees of freedom and varying separately with respect to them (Palatini formalism). Consider the Ricci tensor as a function of the affine connection only. Show that varying the Einstein-Hilbert action w.r.t. the metric, i.e. $\frac{\delta S}{\delta g^{ab}} = 0$, leads to the Einstein field equations.
(Hint: Recall the energy-momentum tensor is found from the matter field Lagrangian $\mathcal{L}_M$ as $T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{ab}}(\sqrt{-g}\mathcal{L}_M)$ and the Jacobi formula $\delta g = g g^{ab}\delta g_{ab}$)

Question 11

Consider a free particle (with mass) moving in a Schwarzschild spacetime, as defined in lecture 4. Assume this particle is parametrised by proper time τ.

(i) Consider particle motion in the equatorial plane. Show that the radial motion $r(\tau)$ of the particle obeys the equation

$$\dot{r}^2 = f(r) = E^2 - \left(1 + \frac{L^2}{r^2}\right)\left(1 - \frac{2M}{r}\right). \tag{15}$$

Circular orbits exist if $f(r) = f^{'}(r) = 0$. Stability requires $f^{''}(r) < 0$ and instability necessitates $f^{''}(r) > 0$.

(ii) Show that particles with mass have stable circular orbits for $r > 6M$.

(iii) Deduce the fractional binding energy $1 - E$ of the innermost stable circular orbit at $r = 6M$ is $1 - \sqrt{8/9}$.

(iv) Show that particles with mass follow circular orbits for $3M < r < 6M$, but that these orbits are unstable.

(v) Show that only photons can have circular orbits at $r = 3M$ (photon sphere). Determine whether this orbit is stable or unstable.

Question 12

As has already been discussed, the singularity at $r = 2M$ for the Schwarzschild spacetime is a coordinate singularity rather than a true physical singularity. As such it may be removed by a suitable coordinate transformation. One such transformation is into Eddington-Finkelstein coordinates, which allows us to follow a trajectory beyond the event horizon and to the singularity at the origin. Consider the tortoise coordinate

$$r^* = r + 2M\ln\left(\frac{r}{2M} - 1\right), \tag{16}$$

along with the in-going and out-going null coordinates

$$u = t - r^*, \quad v = t + r^*. \tag{17}$$

Recast the Schwarzschild solution into Eddington-Finkelstein coordinates and discuss its properties.

Question 13 - Gravitational Waves (Challenging)

We now work in a four-dimensional spacetime. In lecture 4 we discussed the weak field limit and considered a weak gravitational field with spacetime metric

$$g_{ab} = \eta_{ab} + \varepsilon h_{ab} + O(\varepsilon^2), \tag{18}$$

where ε is a small parameter such that terms of second order and higher in ε may be neglected.

(i) Show that

$$g^{ab} = \eta^{ab} - \varepsilon h^{ab} + O(\varepsilon^2). \tag{19}$$

(ii) Show that the fully covariant Riemann curvature tensor is given by

$$R_{abcd} = \frac{1}{2}\varepsilon\big(h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac}\big). \tag{20}$$

(Hint: Start by linearising the definition of the Christoffel symbol)

(iii) Consider the definition $H_{ab} = h_{ab} - \frac{1}{2}h\eta_{ab}$, where $h = h^a{}_a$ and $H = H^a{}_a$. Show that $H = -h$ and therefore $h_{ab} = H_{ab} - \frac{1}{2}H\eta_{ab}$. (This notation makes use of trace-free tensors and any symmetric tensor can be written in trace-free form. Consider $\overline{h}_{ab} = H_{ab}$, the 'bar' operator, which may be applied to any symmetric tensor. In this notation $G_{ab} = \overline{R}_{ab}$ and $\overline{\overline{h}}_{ab} = h_{ab}$.)

(iv) Show that in terms of H_{ab} the Ricci tensor is given as

$$R_{ab} = -\frac{1}{2}\varepsilon\left[\Box H_{ab} - \frac{1}{2}\Box H\eta_{ab} - \partial_b\partial_c H^c{}_a - \partial_a\partial_c H^c{}_b\right], \tag{21}$$

where $\Box H_{ab} = \eta^{cd}\partial_c\partial_d H_{ab}$.

(v) Show that under a guage transformation, H_{ab} transforms as

$$x^c \quad \rightarrow \quad \overline{x}^c = x^c + \varepsilon\zeta^c, \tag{22}$$

$$H_{ab} \quad \rightarrow \quad \overline{H}_{ab} = H_{ab} - \partial_a\zeta_b - \partial_b\zeta_a + \partial_c\zeta^c\eta_{ab}. \tag{23}$$

Deduce that there always exists a new system of coordinates in which $\partial_a\overline{H}^a{}_b = 0$.

(Hint: Consider how the metric transforms under (22) before considering (23))

(vi) Deduce that in this new system the linearised vacuum field equations are given by

$$\Box\overline{h}_{ab} = 0, \tag{24}$$

and the linearised Einstein field equations are given by

$$\Box\overline{h}_{ab} = -16\pi T_{ab}. \tag{25}$$

(vii) Consider a gravitational wave $\overline{h}_{ab}(x^a) = e^{\mathrm{i}k.x}\varepsilon_{ab}$, with $k.x = k_a x^a$ and ε_{ab} the polarisation tensor (symmetric and constant, $\partial_c\varepsilon_{ab} = 0$). How do the linearised vacuum field equations (simple wave equation) translate into a condition on k?

(viii) Deduce further that the gauge choice $\partial_a\overline{H}^a{}_b = 0$ imposes $\varepsilon_{ab}k^a = \frac{1}{2}\varepsilon^a{}_a k_b$, and that there is some remaining gauge freedom $\varepsilon_{ab} \rightarrow \varepsilon_{ab} + k_a v_b + k_b v_a$ for any v_a.

(ix) For a given k_a how may independent elements does the polarisation tensor ε_{ab} contain? Hence determine the number of independent polarisation states for a gravitational wave in a four-dimensional spacetime.

(x) A generalisation of (ix). Can gravitational waves exist in fewer than four dimensions? Can they exist in higher dimensions?

Question 14 - Newtonian Cosmology

Consider the Universe at two points in time, at t_0 and a later time t (now). Consider the scale factor for the Universe, $a(t)$, and density $\rho(t)$.

(i) Assuming a uniform expansion of the universe (Λ=0 and pressure is negligible) determine the kinetic and potential energies, and hence total energy of the Universe, in terms of the scale factor $a(t)$.

(ii) Through consideration of the conservation of energy and mass demonstrate that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{26}$$

where the constant k should be identified and its various values discussed, and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho, \tag{27}$$

the Newtonian cosmological field equations. ($H = \dot{a}/a$ is the Hubble parameter)

(iii) From equations (26) and (27) show that

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0, \tag{28}$$

the conservation equation.

(iii) Integrate the conservation equation to determine the energy density of the Universe. For a flat Universe, with $t_0 = 0$ demonstrate using the Newtonian cosmological field equations and your previous solution that

$$\begin{aligned} a(t) &\sim t^{2/3}, && (29)\\ H &\sim \frac{2}{3t}, && (30)\\ \rho &\sim \frac{1}{t^2}. && (31) \end{aligned}$$

(It follows that the age of the Universe $\sim H^{-1} \propto t$)

Question 15 - General Relativistic Cosmology (Challenging)

Consider the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric given by

$$ds^2 = -dt^2 + a(t)^2 \frac{dx^2 + dy^2 + dz^2}{(1 + k(x^2 + y^2 + z^2)/4)^2}. \tag{32}$$

(i) Compute all non-vanishing Christoffel symbols (8 independent components).

(Hint: You may find it easier to compute the Christoffel symbols directly in this example, rather than via the Lagrangian approach)

(ii) Show that

$$\begin{aligned} R^t{}_t &= 3\frac{\ddot{a}}{a}, && (33)\\ R^x{}_x &= \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}, && (34)\\ R &= 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). && (35) \end{aligned}$$

(iii) Demonstrate an argument to prove that $R_{xx} = R_{yy} = R_{zz}$.

(iv) Consider the energy-momentum tensor

$$T^a_b = \mathrm{diag}(-\rho(t),\ P(t),\ P(t),\ P(t)). \tag{36}$$

Show that the fluid conservation equation, $\nabla_a T^a_b = 0$, leads to one non-trivial result, namely

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0. \tag{37}$$

The result of part (iii) will greatly simplify your calculations. (v) Consider an equation of state for the Universe of the form $P = w\rho$. The most general form of the Einstein field equations is given with cosmological constant as

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}. \tag{38}$$

Using (38) show that the cosmological field equations are given by

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, && (39)\\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(1 + 3w)\rho + \frac{\Lambda}{3}. && (40) \end{aligned}$$

(v) Integrate equation (37) to find the energy density of the Universe. What happens as $w \to 0$? In a flat Universe with zero cosmological constant show that

$$a(t) = \pm\left(\frac{1}{f}\sqrt{\frac{8\pi G}{3}\rho_0}\right)^f (t - t_0)^f, \tag{41}$$

where $f = \frac{2}{3(1+w)}$. For a matter-dominated Universe the pressure is negligible ($w = 0$). For a radiation-dominated Universe $w = 1/3$. Determine how a, H and ρ vary in time. How does this compare to a matter-dominated Universe?