MSSL Advanced Theory Lectures

General Relativity - Problem Sheet 1

Question 1

Which of the following tensor equations are valid, and which are not? Describe the errors in those which are not valid.

$$
\begin{aligned}
K &= R_{abcd}R^{abcd} \\
F_{ab} &= T_{ac}T^{c}{}_{cd} + \frac{1}{4\pi}g^{ab}T_{ab}T^{ab} \\
Q &= g_{ab}U^aU^b
\end{aligned}
$$

Question 2

Consider the type $\binom{2}{0}$ tensor T^{ab} acting in a 4-dimensional manifold (i.e. a, b range from $(0, 1, 2, 3)$, T^{ab} being a 4×4 matrix). How many independent components does T^{ab} have in general? How many independent components do the symmetric and antisymmetric parts $T^{(ab)}$ and $T^{[ab]}$ have? Hence, confirm that a general tensor can be decomposed into a symmetric and an antisymmetric part.

Question 3

Prove the following vector identities

$$
\begin{aligned}
\nabla \times \nabla\phi &= 0, & (1)\\
\nabla.(\nabla \times \overline{a}) &= 0, & (2)\\
\nabla \times (\psi\overline{a}) &= \nabla\psi \times \overline{a} + \psi\nabla \times \overline{a}, & (3)\\
\nabla.(\phi\overline{a}) &= \phi\nabla.\overline{a} + \overline{a}.\nabla\phi, & (4)\\
\nabla \times (\overline{a} \times \overline{b}) &= \overline{a}(\nabla.\overline{b}) - \overline{b}(\nabla.\overline{a}) + (\overline{b}.\nabla)\overline{a} - (\overline{a}.\nabla)\overline{b}. & (5)
\end{aligned}
$$

Question 4

Consider the metric for 3-dimensional flat Euclidean space

$$ds^2 = dx^2 + dy^2 + dz^2. \quad (6)$$

Oblate spheroidal coordinates are related to Cartesian coordinates via the transformation

$$
\begin{aligned}
x &= \sqrt{r^2 + a^2}\sin\theta\cos\phi, & (7)\\
y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi, & (8)\\
z &= r\cos\theta. & (9)
\end{aligned}
$$

Show that the metric in this new coordinate system is given by

$$ds^2 = \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \Delta\sin^2\theta d\phi^2, \quad (10)$$

where

$$
\begin{aligned}
\Sigma &= r^2 + a^2\cos^2\theta, & (11)\\
\Delta &= r^2 + a^2. & (12)
\end{aligned}
$$

What happens as a$\rightarrow$0?

Question 5
Solve the geodesic equations of motion for the following two metrics

$$ds^2 = dx^2 + dy^2 + dz^2, \qquad (13)$$
$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2. \qquad (14)$$

What do the solutions imply? Compute the non-zero Christoffel symbols for (14).

Question 6
Show that the line element

$$ds^2 = g_{ab} dX^a dX^b, \qquad (15)$$

is a scalar, i.e. determine how ds^2 transforms under general coordinate transformations.
Note: It may prove easier to use the following notation for transformation laws for contravariant and covariant vectors

$$V^{a'} = \frac{\partial X^{a'}}{\partial X^a} V^a, \qquad (16)$$
$$W_{b'} = \frac{\partial X^b}{\partial X^{b'}} W_b, \qquad (17)$$

from now on, which can easily be generalised to tensors. This notation will simplify the working in later questions.
(Hint: Consider how $g_{a'b'} dX^{a'} dX^{b'}$ transforms under general coordinate transformations)

Question 7
Demonstrate how

$$A^b{}_{,a} \equiv \partial_a A^b$$

transforms under general coordinate transformations. Is $A^b{}_{,a}$ a tensor? In General Relativity, as in tensor analysis, what property must a good operator possess? Should this be used as a partial derivative operator in General Relativity?

Question 8 (Challenging)
Recall the Christoffel symbol from the lectures, also known as the affine connection. We have already mentioned in lecture 2 that in moving from flat space to General Relativity and curved spacetimes

$$\partial_a \rightarrow \nabla_a,$$

namely that the partial derivative now becomes a *covariant* derivative operator. Given the result of question 7, the covariant derivative operator ∇_a (being inherently covariant) must be composed of the partial derivative operator ∂_a and a correction term (given spacetime is no longer flat), which can be expressed in terms of the affine connection $\Gamma^a{}_{bc}$, namely

$$\nabla_a V^b = \partial_a V^b + \Gamma^b{}_{ac} V^c. \qquad (18)$$

Using the above expression for the covariant derivative, show that the Christoffel symbol transforms as

$$\Gamma^{a'}{}_{b'c'} = \frac{\partial X^c}{\partial X^{c'}} \frac{\partial X^b}{\partial X^{b'}} \frac{\partial X^{a'}}{\partial X^a} \Gamma^a_{bc} - \frac{\partial X^b}{\partial X^{b'}} \frac{\partial X^c}{\partial X^{c'}} \frac{\partial^2 X^{a'}}{\partial X^b \partial X^c}. \qquad (19)$$

What does this tell us about the Christoffel symbol, and why?

Question 9

In General Relativity the most general static, spherically symmetric metric is given by

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{20}$$

Compute the non-zero Christoffel symbols for this metric (9 in total).

Question 10

Working with the Minkowski signature convention $(+,-,-,-)$, for a given scalar potential ϕ and vector potential $\overline{A}$ the 4-potential A^a can be defined as $A^a = \begin{pmatrix} \phi \\ \mathrm{A} \end{pmatrix}$, $A_a = \eta_{ab}A^b = (\phi, -\overline{\mathrm{A}})$. From the definition of the magnetic and electric field

$$\overline{\mathrm{B}} = \nabla \times \overline{\mathrm{A}}, \tag{21}$$

$$\overline{\mathrm{E}} = -\nabla\phi - \partial_t \overline{\mathrm{A}}, \tag{22}$$

and the Faraday tensor

$$F_{ab} = A_{b,a} - A_{a,b}, \tag{23}$$

show explicitly that in Cartesian coordinates the Faraday tensor is given by

$$F_{ab} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}, \tag{24}$$

where (a,b) run over (t,x,y,z). The dual Faraday tensor is defined as

$$*F^{ab} = \frac{1}{2}\varepsilon^{abcd}F_{cd}, \tag{25}$$

where ε^{abcd} is the Levi-Civita symbol, defined in lecture 1. Show explicitly that

$$*F^{ab} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix}. \tag{26}$$

Furthermore, show that

$$*F_{ab}F^{ab} = 4\,(\overline{\mathrm{B}}.\overline{\mathrm{E}}), \tag{27}$$

$$F_{ab}F^{ab} = 2(\mathrm{B}^2 - \mathrm{E}^2). \tag{28}$$

If you recall how to find the determinant of a 4×4 matrix then show that

$$\det(F) = (\overline{\mathrm{B}}.\overline{\mathrm{E}})^2, \tag{29}$$

which also holds for the dual Faraday tensor. (Hint: $F^{ab} \leftrightarrow F_{ab} \implies \overline{\mathrm{E}} \leftrightarrow -\overline{\mathrm{E}}$, $\overline{\mathrm{B}}$ fixed)