Hydrodynamics & Magnetohydrodynamics: Solutions to Exercises in Lecture III

Ziri Younsi

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Exercise 1

Show that the scalar quantity d^3p/p^0 is a Lorentz invariant, where $\mathbf{p} = cm\mathbf{u}$ is the four-momentum and $\mathbf{u}$ the four-velocity. [Hint: exploit the normalisation condition of the four-velocity].

Solution 1

Starting from

$$\vec{p} = cm\vec{u}\,, \tag{1.1}$$

the contravariant and covariant components of the 4-momentum may be written

$$\begin{aligned} p^\mu &= cm\gamma\left(1, v^i\right)\,, &(1.2)\\ p_\mu &= cm\gamma\left(-1, v^i\right)\,, &(1.3)\end{aligned}$$

where $\gamma = (1-v^2)^{-1/2}$ is the Lorentz factor. From this we calculate the inner-product of the 4-momentum as

$$\begin{aligned} p^2 &= p^\mu p_\mu \\ &= c^2m^2\gamma^2\left(v^2-1\right) \\ &= -m^2c^2\,. &(1.4)\end{aligned}$$

This may also be calculated directly from principles of covariance and the normalisation condition of the 4-velocity ($u^\mu u_\mu = -1$), yielding

$$\begin{aligned} p^2 &= p^\mu p_\mu \\ &= c^2m^2\left(u^\mu u_\mu\right) \\ &= -m^2c^2\,. &(1.5)\end{aligned}$$

Next, consider a Lorentz boost in the x-direction, which may be written as

$$\Lambda^{\nu'}{}_{\mu} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{1.6}$$

We may transform between reference frames using

$$p^{\nu'} = \Lambda^{\nu'}{}_{\mu} p^{\mu} , \tag{1.7}$$

from which the following transformations are obtained

$$p^{0'} = \gamma \left(p^0 - vp^1\right) , \tag{1.8}$$

$$p^{1'} = \gamma \left(p^1 - vp^0\right) , \tag{1.9}$$

$$p^{2'} = p^2 , \tag{1.10}$$

$$p^{3'} = p^3 . \tag{1.11}$$

Now consider the momentum space volume element, which may be written in two different frames, related through a Lorentz transformation, as $d^3p = dp^1\, dp^2\, dp^3$ and $d^3p' = dp^{1'}\, dp^{2'}\, dp^{3'}$. Differentiating equation (1.9) with respect to p^1 yields

$$\frac{dp^{1'}}{dp^1} = \gamma \left(1 - v\frac{dp^0}{dp^1}\right) . \tag{1.12}$$

Separating $p^{\mu}p_{\mu} = -m^2c^2$ into temporal and spatial parts (assuming [−,+,+,+] signature) yields $-p^0p_0 + p^ip_i = -m^2c^2$. Since we are working in special relativity, $p^0p_0 = \left(p^0\right)^2$ and $p^ip_i = \left(p^i\right)^2$ and we obtain

$$p^0 = \left[\left(p^i\right)^2 + m^2c^2\right]^{1/2} . \tag{1.13}$$

Differentiating p^0 with respect to p^1 yields

$$\begin{aligned} \frac{dp^0}{dp^1} &= p^1 \left[\left(p^i\right)^2 + m^2c^2\right]^{-1/2} \\ &= \frac{p^1}{p^0} , \end{aligned} \tag{1.14}$$

and thus we may write

$$\begin{aligned} \frac{dp^{1'}}{dp^1} &= \gamma \left(1 - v\frac{p^1}{p^0}\right) \\ &= \frac{\gamma \left(p^0 - vp^1\right)}{p^0} \\ &= \frac{p^{0'}}{p^0} . \end{aligned} \tag{1.15}$$

Since $dp^{2'} = dp^2$ and $dp^{3'} = dp^3$ we find

$$\begin{aligned} d^3p' &= dp^{1'}\, dp^{2'}\, dp^{3'} \\ &= \frac{p^{0'}}{p^0} dp^1\, dp^2\, dp^3 , \end{aligned} \tag{1.16}$$

from which it immediately follows that

$$\frac{d^3p'}{p^{0'}} = \frac{d^3p}{p^0} , \tag{1.17}$$

hence d^3p/p^0 is a Lorentz invariant, as required.

Exercise 2

Show that the conservation equation for the total energy density

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{\mathbf{v}}^2 + \rho\epsilon\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho\vec{\mathbf{v}}^2 + \rho\epsilon + p\right)\vec{\mathbf{v}}\right] = \frac{\rho}{m}\vec{\mathbf{F}}\cdot\vec{\mathbf{v}}\,, \tag{2.1}$$

can also be written as

$$\frac{D}{Dt}\left(\frac{1}{2}\rho\vec{\mathbf{v}}^2 + \rho\epsilon\right) + \left(\frac{1}{2}\rho\vec{\mathbf{v}}^2 + \rho\epsilon + p\right)\vec{\nabla}\cdot\vec{\mathbf{v}} = \rho\vec{\mathbf{v}}\cdot\left(\frac{\vec{\mathbf{F}}}{m} - \frac{1}{\rho}\vec{\nabla}p\right)\,, \tag{2.2}$$

where D/Dt is the Lagrangian derivative.

Solution 2

Let us first define the shorthand notation $w \equiv \frac{1}{2}\rho\vec{\mathbf{v}}^2 + \rho\epsilon$. We may thus write the second term in equation (2.1) as

$$\begin{aligned}\vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho\vec{\mathbf{v}}^2 + \rho\epsilon + p\right)\vec{\mathbf{v}}\right] &= \vec{\mathbf{v}}\cdot\vec{\nabla}(w+p) + (w+p)\,\vec{\nabla}\cdot\vec{\mathbf{v}} \\ &= \vec{\mathbf{v}}\cdot\vec{\nabla}w + (w+p)\,\vec{\nabla}\cdot\vec{\mathbf{v}} + \vec{\mathbf{v}}\cdot\vec{\nabla}p \end{aligned} \tag{2.3}$$

Equation (2.1) may now be written as

$$\frac{\partial w}{\partial t} + \vec{\mathbf{v}}\cdot\vec{\nabla}w + (w+p)\,\vec{\nabla}\cdot\vec{\mathbf{v}} + \vec{\mathbf{v}}\cdot\vec{\nabla}p = \frac{\rho}{m}\vec{\mathbf{F}}\cdot\vec{\mathbf{v}}\,, \tag{2.4}$$

which upon rearrangement yields

$$\left(\frac{\partial}{\partial t} + \vec{\mathbf{v}}\cdot\vec{\nabla}\right)w + (w+p)\,\vec{\nabla}\cdot\vec{\mathbf{v}} = \frac{\rho}{m}\vec{\mathbf{F}}\cdot\vec{\mathbf{v}} - \vec{\mathbf{v}}\cdot\vec{\nabla}p\,, \tag{2.5}$$

Introducing the Lagrangian derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{\mathbf{v}}\cdot\vec{\nabla}$, we finally obtain

$$\frac{Dw}{Dt} + (w+p)\,\vec{\nabla}\cdot\vec{\mathbf{v}} = \rho\vec{\mathbf{v}}\cdot\left(\frac{\vec{\mathbf{F}}}{m} - \frac{1}{\rho}\vec{\nabla}p\right)\,, \tag{2.6}$$

which is precisely equation (2.2), as required.

Exercise 3

Optional. Consider a two-dimensional flow in which two fluids of the same type have uniform velocity in opposite directions and are subject to an external gravitational potential with uniform acceleration g and uniform pressure p. Determine the evolution of the fluid when perturbed; compare your results with the properties of the Kelvin-Helmholtz instability. [Hint: Use a Cartesian coordinate system in which the fluids have velocities $\vec{\mathbf{v}}_1 = (v^x, 0)$, $\vec{\mathbf{v}}_2 = (-v^x, 0)$, and introduce perturbations in velocity and pressure, i.e., $\vec{\mathbf{v}}_1 \to \vec{\mathbf{v}}_1 = (v^x + \delta v^x, \delta v^y)$, $\vec{\mathbf{v}}_2 \to \vec{\mathbf{v}}_2 = (-v^x + \delta v^x, \delta v^y)$. Study the space of solutions of the linearised equations].

Solution 3