Hydrodynamics & Magnetohydrodynamics: Solutions to Exercises in Lecture II

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Exercise 1

Using the following ansatz for the Maxwell-Boltzmann (equilibrium) distribution function

$$\ln\left[f_0\left(\vec{\mathbf{u}}\right)\right] = -\mathcal{A}\left(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\right)^2 + \ln\mathcal{C}\,, \tag{1.1}$$

and the definition of the specific internal energy

$$\begin{aligned} \epsilon &:= \frac{1}{2}\Big\langle|\vec{\mathbf{u}} - \vec{\mathbf{v}}|^2\Big\rangle & (1.2)\\ &= \frac{1}{2n}\int d^3u\,|\vec{\mathbf{u}} - \vec{\mathbf{v}}|^2\,f\,, & (1.3)\end{aligned}$$

prove that the constants $\mathcal{A}$ and $\mathcal{C}$ are given by

$$\begin{aligned} \mathcal{A} &= \frac{3}{4\epsilon}\,, & (1.4)\\ \mathcal{C} &= n\left(\frac{3}{4\pi\epsilon}\right)^{3/2}\,. & (1.5)\end{aligned}$$

Solution 1

Exponentiating both sides of equation (1.1) yields

$$f_0\left(\vec{\mathbf{u}}\right) = \mathcal{C}\exp\left[-\mathcal{A}\left(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\right)^2\right]\,. \tag{1.6}$$

The number density is defined as

$$\begin{aligned} n &= \int d^3u\,f_0 \\ &= \mathcal{C}\int_{-\infty}^{\infty} d^3u\,\exp\left[-\mathcal{A}\left(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\right)^2\right]\,. & (1.7)\end{aligned}$$

Introducing the change of variable $\vec{\mathbf{v}} = \vec{\mathbf{u}} - \vec{\mathbf{u}}_0$, i.e. $d\vec{\mathbf{v}} = d\vec{\mathbf{u}}$, the above integral becomes

$$\begin{aligned} n &= \mathcal{C}\int_{-\infty}^{\infty} d^3v\,\exp\left(-\mathcal{A}\vec{\mathbf{v}}^2\right) \\ &= \mathcal{C}\left(\frac{\pi}{\mathcal{A}}\right)^{3/2}\,, & (1.8)\end{aligned}$$

and thus we obtain

$$\mathcal{C} = n\left(\frac{\mathcal{A}}{\pi}\right)^{3/2} . \tag{1.9}$$

From the definition of the specific internal energy we obtain

$$\begin{aligned}
\epsilon &= \frac{1}{2}\left\langle|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0|^2\right\rangle \\
&= \frac{1}{2}\left\langle|\vec{\mathbf{v}}|^2\right\rangle \\
&= \frac{\mathcal{C}}{2n}\int d^3v\,|\vec{\mathbf{v}}|^2\,\exp\left(-\mathcal{A}\vec{\mathbf{v}}^2\right) \\
&= \frac{\mathcal{C}}{2n}\left(\frac{3}{2}\frac{\pi^{3/2}}{\mathcal{A}^{5/2}}\right) .
\end{aligned} \tag{1.10}$$

Substituting the value already obtained for $\mathcal{C}$ from equation (1.9) into equation (1.10) yields

$$\begin{aligned}
\epsilon &= \left(\frac{1}{2n}\right)\left[n\left(\frac{\mathcal{A}}{\pi}\right)^{3/2}\right]\left(\frac{3}{2}\frac{\pi^{3/2}}{\mathcal{A}^{5/2}}\right) \\
&= \frac{3}{4\mathcal{A}} .
\end{aligned} \tag{1.11}$$

It immediately follows that

$$\mathcal{A} = \frac{3}{4\epsilon} , \tag{1.12}$$

and from equation (1.9) it therefore follows that

$$\mathcal{C} = n\left(\frac{3}{4\pi\epsilon}\right)^{3/2} , \tag{1.13}$$

as required.

Exercise 2

Recalling that for a classical monoatomic fluid the specific internal energy is given by

$$\epsilon = \frac{3}{2}\frac{k_{\mathrm{B}}T}{m} , \tag{2.1}$$

show that the explicit expression for the Maxwell-Boltzmann distribution function is

$$f_0\left(\vec{\mathbf{u}}\right) = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\exp\left(-\frac{m\left(\vec{\mathbf{u}} - \vec{\mathbf{v}}\right)^2}{2k_{\mathrm{B}}T}\right) . \tag{2.2}$$

Solution 2

In the solution to Exercise 1, the values of $\mathcal{A}$ and $\mathcal{C}$ in terms of ϵ were derived. Substituting the value of ϵ into these expressions yields

$$\mathcal{A} = \frac{m}{2k_{\mathrm{B}}T} \tag{2.3}$$

$$\mathcal{C} = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2} . \tag{2.4}$$

Substituting these expressions into the general form of the Maxwell-Boltzmann distribution function given in Exercise 1 yields

$$f_0\left(\vec{\mathbf{u}}\right) = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2} \exp\left(-\frac{m\left(\vec{\mathbf{u}}-\vec{\mathbf{v}}\right)^2}{2k_{\mathrm{B}}T}\right) , \tag{2.5}$$

as required.

Exercise 3

Using the definition of the Maxwell-Boltzmann distribution function for the velocity norm u for a fluid with zero macroscopic velocity (i.e., $\vec{\mathbf{v}} = 0$)

$$f_0\left(u\right) = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2} \exp\left(-\frac{mu^2}{2k_{\mathrm{B}}T}\right) , \tag{3.1}$$

show that the average speed is

$$v = \left(\frac{8k_{\mathrm{B}}T}{\pi m}\right)^{1/2} . \tag{3.2}$$

Solution 3

We begin the exercise by noting that in velocity space we may adopt a spherical polar coordinate system (r, θ, ϕ), through which the Cartesian coordinate system is related as

$$du_x\, du_y\, du_z = u^2 \sin\theta\, du\, d\theta\, d\phi\, . \tag{3.3}$$

Starting from the Maxwell-Boltzmann distribution with zero macroscopic velocity, the average speed is calculated as

$$\begin{aligned} v &= \langle u \rangle \\ &= \frac{1}{n}\int d^3u\, \vec{\mathbf{u}}\, f_0(\vec{\mathbf{u}}) \\ &= \frac{1}{n}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta \int_0^{\infty} du\, u^3 f_0(u) \\ &= \frac{4\pi}{n}\int_0^{\infty} du\, u^3 f_0(u) \\ &= 4\pi\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2} \int_0^{\infty} du\, u^3 \exp\left(-\frac{mu^2}{2k_{\mathrm{B}}T}\right) . \end{aligned} \tag{3.4}$$

This is equivalent to writing

$$\begin{aligned} v &= \frac{1}{n}\int_0^\infty du\, u \left[4\pi n \left(\frac{m}{2\pi k_\mathrm{B} T}\right)^{3/2} u^2 \exp\left(-\frac{mu^2}{2k_\mathrm{B}T}\right)\right] \\ &= \frac{1}{n}\int_0^\infty du\, u\, f(u)\,, \end{aligned} \tag{3.5}$$

thus the Maxwell-Boltzmann distribution with zero macroscopic velocity is

$$f(u) = 4\pi n \left(\frac{m}{2\pi k_\mathrm{B} T}\right)^{3/2} u^2 \exp\left(-\frac{mu^2}{2k_\mathrm{B}T}\right)\,. \tag{3.6}$$

To proceed further, we introduce the change of variable $w = u^2$, i.e. $du = dw/(2u)$, which yields

$$v = 2\pi \left(\frac{m}{2\pi k_\mathrm{B} T}\right)^{3/2} \int_0^\infty dw\, w \exp\left(-\frac{mw}{2k_\mathrm{B}T}\right)\,. \tag{3.7}$$

Integrating the above expression by parts yields

$$\begin{aligned} v &= 2\pi \left(\frac{m}{2\pi k_\mathrm{B} T}\right)^{3/2} \left\{\left[-\left(\frac{2k_\mathrm{B}T}{m}\right) w\, \exp\left(-\frac{mw}{2k_\mathrm{B}T}\right)\right]_0^\infty + \left(\frac{2k_\mathrm{B}T}{m}\right)\int_0^\infty dw\, \exp\left(-\frac{mw}{2k_\mathrm{B}T}\right)\right\} \\ &= \left(\frac{4}{\pi}\right)^{1/2} \left(\frac{m}{2k_\mathrm{B}T}\right)^{1/2} \int_0^\infty dw\, \exp\left(-\frac{mw}{2k_\mathrm{B}T}\right) \\ &= \left(\frac{4}{\pi}\right)^{1/2} \left(\frac{m}{2k_\mathrm{B}T}\right)^{1/2} \left(\frac{2k_\mathrm{B}T}{m}\right) \left[-\exp\left(-\frac{mw}{2k_\mathrm{B}T}\right)\right]_0^\infty \\ &= \left(\frac{8k_\mathrm{B}T}{\pi m}\right)^{1/2}\,, \end{aligned} \tag{3.8}$$

as required.

Exercise 4

Optional. Show that the most probable speed is

$$v = \left(\frac{2k_\mathrm{B}T}{m}\right)^{1/2}\,. \tag{4.1}$$

Solution 4

The most probable speed is just the maximum of the Maxwell-Boltzmann distribution function. Recall that in Exercise 3 the Maxwell-Boltzmann distribution with zero macroscopic velocity in polar coordinates was found as

$$f(u) = 4\pi n \left(\frac{m}{2\pi k_\mathrm{B} T}\right)^{3/2} u^2 \exp\left(-\frac{mu^2}{2k_\mathrm{B}T}\right)\,. \tag{4.2}$$

We may thus find the maximum by differentiating $f(u)$. Using the definitions of $\mathcal{A}$ and $\mathcal{C}$ obtained in Solution 2 we obtain the most probable speed, $u_{\rm p}$, by solving the following equation

$$\begin{aligned}
\left.\frac{df(u)}{du}\right|_{u=u_{\rm p}} &= \frac{d}{du}\left\{4\pi n\left(\frac{m}{2\pi k_{\rm B}T}\right)^{3/2}\left[u^2\exp\left(-\frac{mu^2}{2k_{\rm B}T}\right)\right]\right\}\Bigg|_{u=u_{\rm p}} \\
&= \frac{d}{du}\left\{4\pi\,\mathcal{C}\left[u^2\exp\left(-\mathcal{A}\,u^2\right)\right]\right\}\Big|_{u=u_{\rm p}} \\
&= 4\pi\,\mathcal{C}\left(2u_{\rm p}-2\mathcal{A}\,u_{\rm p}^3\right)\exp\left(-\mathcal{A}\,u_{\rm p}^2\right) \\
&= 8\pi\,\mathcal{C}u_{\rm p}\left(1-\mathcal{A}\,u_{\rm p}^2\right)\exp\left(-\mathcal{A}\,u_{\rm p}^2\right)=0\,.
\end{aligned} \tag{4.3}$$

Neglecting the trivial solution $u_{\rm p}=0$, we obtain the most probable speed as

$$\begin{aligned}
u_{\rm p} &= \left(\frac{1}{\mathcal{A}}\right)^{1/2} \\
&= \left(\frac{2k_{\rm B}T}{m}\right)^{1/2}\,,
\end{aligned} \tag{4.4}$$

as required.