General Relativity: Solutions to exercises in Lecture IX

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Winter Semester 2015

Exercise 1

Consider a torus in a two-dimensional Euclidean space described by the spherical co-ordinate system (θ, ϕ). The line element of the torus is then given by

$$\mathrm{d}s^2 = (b + a\sin\phi)^2\,\mathrm{d}\theta^2 + a^2\mathrm{d}\phi^2 \ , \tag{1}$$

where b and a denote the torus radius and the radius of its section, respectively.

Compute the Christoffel symbol components and the non-vanishing components of the (Riemann) curvature tensor. (Hint: remember that there is only one linearly independent component of the Riemann tensor in a spacetime of dimension 2).

Solution 1

First let us consider the components of the metric and their partial derivatives:

$$g_{\mu\nu} = \begin{pmatrix} (b + a\sin\phi)^2 & 0 \\ 0 & a^2 \end{pmatrix} \ , \tag{2}$$

$$g^{\mu\nu} = \begin{pmatrix} (b + a\sin\phi)^{-2} & 0 \\ 0 & a^{-2} \end{pmatrix} \ , \tag{3}$$

$$g_{\mu\nu,\theta} = 0 \ , \tag{4}$$

$$g_{\mu\nu,\phi} = \begin{pmatrix} 2a\,(b + a\sin\phi)\cos\phi & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{5}$$

Next, recall the definition of the Christoffel symbols:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta} \right) \ . \tag{6}$$

Since α can only be θ or ϕ and the metric is diagonal, we may proceed as follows:

$$
\begin{aligned}
\Gamma^{\theta}_{\beta\gamma} &= \frac{1}{2} g^{\theta\theta} \left(g_{\theta\beta,\gamma} + g_{\gamma\theta,\beta} - \cancel{g_{\beta\gamma,\theta}} \right) \\
&= \frac{1}{2} g^{\theta\theta} g_{\theta\theta,\phi} && (7) \\
&= \frac{a\cos\phi}{b + a\sin\phi} && (8) \\
\Gamma^{\phi}_{\beta\gamma} &= \frac{1}{2} g^{\phi\phi} \left(\cancel{g_{\phi\beta,\gamma}} + \cancel{g_{\gamma\phi,\beta}} - g_{\beta\gamma,\phi} \right) \\
&= -\frac{1}{2} g^{\phi\phi} g_{\theta\theta,\phi} && (9) \\
&= -\frac{(b + a\sin\phi)\cos\phi}{a} \ . && (10)
\end{aligned}
$$

It immediately follows that the only non-zero Christoffel symbols are given by:

$$
\begin{aligned}
\Gamma^{\theta}_{\theta\phi} &= \frac{a\cos\phi}{b + a\sin\phi} \ , && (11) \\
\Gamma^{\phi}_{\theta\theta} &= -\frac{(b + a\sin\phi)\cos\phi}{a} \ . && (12)
\end{aligned}
$$

Next, recall the definition of the Riemann tensor:

$$
R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\mu}_{\nu\alpha,\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\nu\alpha} \ , \tag{13}
$$

which may also be written more compactly as

$$
R^{\mu}{}_{\nu\alpha\beta} = \left(\Gamma^{\mu}_{\nu\beta,\alpha} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\nu\beta} \right) - (\alpha \leftrightarrow \beta) \ , \tag{14}
$$

where $(\alpha \leftrightarrow \beta)$ denotes writing down the first term in brackets with α and β exchanged. Looking at the first term in equation (13), we know that $\Gamma^{\mu}_{\nu\beta,\alpha}$ is non-zero only if $\alpha = \phi$ (partial derivative is non-zero). Next, we are free to choose (μ, ν, β) such that the Christoffel symbol is also non-zero. This yields the choices $(\mu, \nu, \beta) = (\theta, \theta, \phi)$ or (ϕ, θ, θ). Let us take $(\mu, \nu, \beta) = (\theta, \theta, \phi)$, which yields

$$
\begin{aligned}
R^{\theta}{}_{\theta\phi\phi} &= \Gamma^{\theta}_{\theta\phi,\phi} + \Gamma^{\theta}_{\rho\phi}\Gamma^{\rho}_{\theta\phi} - \Gamma^{\theta}_{\theta\phi,\phi} - \Gamma^{\theta}_{\rho\phi}\Gamma^{\rho}_{\theta\phi} \\
&= \Gamma^{\theta}_{\theta\phi,\phi} + \Gamma^{\theta}_{\theta\phi}\Gamma^{\theta}_{\theta\phi} + \cancel{\Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi}} - \Gamma^{\theta}_{\theta\phi,\phi} - \Gamma^{\theta}_{\theta\phi}\Gamma^{\theta}_{\theta\phi} - \cancel{\Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi}} \\
&= \Gamma^{\theta}_{\theta\phi,\phi} + \left(\Gamma^{\theta}_{\theta\phi}\right)^2 - \Gamma^{\theta}_{\theta\phi,\phi} - \left(\Gamma^{\theta}_{\theta\phi}\right)^2 \\
&= 0 \ . && (15)
\end{aligned}
$$

Instead, let us now try $\alpha = \theta$ and $\beta = \phi$ in equation (13). We obtain

$$
R^{\mu}{}_{\nu\theta\phi} = \cancel{\Gamma^{\mu}_{\nu\phi,\theta}} + \Gamma^{\mu}_{\rho\theta}\Gamma^{\rho}_{\nu\phi} - \Gamma^{\mu}_{\nu\theta,\phi} - \Gamma^{\mu}_{\rho\phi}\Gamma^{\rho}_{\nu\theta} \ . \tag{16}
$$

Next, let us ensure the partial derivative of the Christoffel symbol does not vanish by choosing $\mu = \theta$ and $\nu = \phi$, which yields:

$$
\begin{aligned}
R^{\theta}{}_{\phi\theta\phi} &= \Gamma^{\theta}_{\rho\theta}\cancel{\Gamma^{\rho}_{\phi\phi}} - \Gamma^{\theta}_{\phi\theta,\phi} - \Gamma^{\theta}_{\rho\phi}\Gamma^{\rho}_{\phi\theta} \\
&= -\Gamma^{\theta}_{\phi\theta,\phi} - \Gamma^{\theta}_{\theta\phi}\Gamma^{\theta}_{\phi\theta} \\
&= \frac{a\,(a + b\sin\phi)}{(b + a\sin\phi)^2} - \frac{a^2\cos^2\phi}{(b + a\sin\phi)^2} \\
&= \frac{a\sin\phi}{(b + a\sin\phi)} \ . && (17)
\end{aligned}
$$

Consequently we may calculate the (only) non-vanishing component of the fully-covariant Riemann curvature tensor as:

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} \\ &= a \sin\phi \, (b + a \sin\phi) \ . \end{aligned} \tag{18}$$

Exercise 2

Consider the two-dimensional spacetime with line element

$$\mathrm{d}s^2 = \mathrm{d}v^2 - v^2 \mathrm{d}u^2 \ . \tag{19}$$

Compute the Christoffel symbols and the non-vanishing components of the curvature tensor.

Solution 2

As in question 1, let us first start by writing down the metric components and their partial derivatives. First consider the components of the metric and their partial derivatives:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -v^2 \end{pmatrix} \ , \tag{20}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -v^2 \end{pmatrix} \ , \tag{21}$$

$$g_{\mu\nu,v} = \begin{pmatrix} 0 & 0 \\ 0 & -2v \end{pmatrix} \ , \tag{22}$$

$$g_{\mu\nu,u} = 0 \ . \tag{23}$$

Next, recall the definition of the Christoffel symbols:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta} \right) \ . \tag{24}$$

We may calculate the Christoffel symbols as before:

$$\begin{aligned} \Gamma^{v}_{\beta\gamma} &= \frac{1}{2} g^{vv} \left(\cancel{g_{v\beta,\gamma}} + \cancel{g_{\gamma v,\beta}} - g_{\beta\gamma,v} \right) \\ &= -\frac{1}{2} g^{vv} g_{\beta\gamma,v} \\ &= -\frac{1}{2} g^{vv} g_{uu,v} \\ &= -\frac{1}{2} (1)(-2v) \\ &= v \ , \end{aligned} \tag{25}$$

$$
\begin{aligned}
\Gamma^{u}_{\beta\gamma} &= \frac{1}{2}g^{uu}\left(g_{u\beta,\gamma} + g_{\gamma u,\beta} - \cancel{g_{\beta\gamma,u}}\right) \\
&= \frac{1}{2}g^{uu}\left(g_{u\beta,\gamma} + g_{\gamma u,\beta}\right) \\
&= \frac{1}{2}g^{uu}g_{uu,v} \\
&= \frac{1}{2}\left(-\frac{1}{v^2}\right)(-2v) \\
&= \frac{1}{v} \ . \qquad (26)
\end{aligned}
$$

Thus we obtain the only non-zero Christoffel symbols as:

$$
\begin{aligned}
\Gamma^{v}_{uu} &= v \ , \qquad (27) \\
\Gamma^{u}_{uv} &= \frac{1}{v} \ . \qquad (28)
\end{aligned}
$$

Next recall the Riemann curvature tensor as defined in equation (13). Let us first make the first term vanish by choosing $\alpha = u$, yielding

$$
R^{\mu}{}_{\nu u\beta} = \cancel{\Gamma^{\mu}_{\nu\beta,u}} + \Gamma^{\mu}_{\rho u}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\mu}_{\nu u,\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\nu u} \ . \qquad (29)
$$

Now let us expand the sum over the dummy indices ρ:

$$
R^{\mu}{}_{\nu u\beta} = \Gamma^{\mu}_{vu}\Gamma^{v}_{\nu\beta} + \Gamma^{\mu}_{uu}\Gamma^{u}_{\nu\beta} - \Gamma^{\mu}_{\nu u,\beta} - \Gamma^{\mu}_{v\beta}\Gamma^{v}_{\nu u} - \Gamma^{\mu}_{u\beta}\Gamma^{u}_{\nu u} \ . \qquad (30)
$$

Next, let us focus on ensuring the $\Gamma^{\mu}_{\nu u,\beta}$ term is non-vanishing, which requires us to set $\beta = v$:

$$
R^{\mu}{}_{\nu uv} = \Gamma^{\mu}_{vu}\Gamma^{v}_{\nu v} + \Gamma^{\mu}_{uu}\Gamma^{u}_{\nu v} - \Gamma^{\mu}_{\nu u,v} - \cancel{\Gamma^{\mu}_{vv}}\Gamma^{v}_{\nu u} - \Gamma^{\mu}_{uv}\Gamma^{u}_{\nu u} \ . \qquad (31)
$$

From equation (31) let us first consider $\mu = u$ and $\nu = v$:

$$
\begin{aligned}
R^{u}{}_{vuv} &= \Gamma^{u}_{vu}\cancel{\Gamma^{v}_{vv}} + \cancel{\Gamma^{u}_{uu}\Gamma^{u}_{vv}} - \Gamma^{u}_{vu,v} - \Gamma^{u}_{uv}\Gamma^{u}_{vu} \\
&= -\Gamma^{u}_{vu,v} - \left(\Gamma^{u}_{uv}\right)^2 \\
&= \frac{1}{v^2} - \frac{1}{v^2} \\
&= 0 \ . \qquad (32)
\end{aligned}
$$

Let us next (and finally) consider the case where $\mu = v$ and $\nu = u$:

$$
\begin{aligned}
R^{v}{}_{uuv} &= \cancel{\Gamma^{v}_{vu}\Gamma^{v}_{uv}} + \Gamma^{v}_{uu}\Gamma^{u}_{uv} - \Gamma^{v}_{uu,v} - \cancel{\Gamma^{v}_{uv}\Gamma^{u}_{uu}} \\
&= \Gamma^{v}_{uu}\Gamma^{u}_{uv} - \Gamma^{v}_{uu,v} \\
&= v\left(\frac{1}{v}\right) - 1 \\
&= 0 \ . \qquad (33)
\end{aligned}
$$

We may conclude that for this particular spacetime the Riemann tensor vanishes everywhere. As such, we may say that our spacetime is flat.

Exercise 3

Consider a geodesic curve $\mathcal{C}$ and its tangent vector $\mathbf{V}$. Compute the expression for the second convective derivative of a vector field $\mathbf{A}$ along $\mathcal{C}$, i.e. an explicit expression in component form of

$$\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\mathbf{A} \ . \tag{34}$$

Recast the resulting expressions in terms of tensors that you have already encountered and interpret the results.

Solution 3

First we must calculate the action of the convective derivative on $\mathbf{A}$:

$$\begin{aligned}\nabla_{\mathbf{V}}\mathbf{A} &= V^\mu \nabla_\mu A^\alpha \\ &= V^\mu \left(\partial_\mu A^\alpha + \Gamma^\alpha_{\mu\beta} A^\beta\right) \ . \end{aligned} \tag{35}$$

At this point it is important to remark that equation (35) is actually a rank-1 contravariant tensor which we may call T^α (all other indices are dummy indices). With this in mind, we may write the second convective derivative of $\mathbf{A}$ as

$$\begin{aligned}\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\mathbf{A} &= \nabla_{\mathbf{V}}\left(\nabla_{\mathbf{V}}\mathbf{A}\right) \\ &= V^\nu \nabla_\nu T^\alpha \\ &= V^\nu \left(\partial_\nu T^\alpha + \Gamma^\alpha_{\nu\rho} T^\rho\right) \ . \end{aligned} \tag{36}$$

From here we must explicitly expand (36), yielding:

$$\begin{aligned}\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\mathbf{A} = \ & V^\nu\Big[\left(\partial_\nu V^\mu\right)\left(\partial_\mu A^\alpha\right) + V^\mu \partial_\nu\partial_\mu A^\alpha + \left(\partial_\nu V^\mu\right)\Gamma^\alpha_{\mu\beta}A^\beta \\ & + V^\mu \partial_\nu \Gamma^\alpha_{\mu\beta} A^\beta + V^\mu \Gamma^\alpha_{\mu\beta}\partial_\nu A^\beta \\ & + \Gamma^\alpha_{\nu\rho} V^\mu \partial_\mu A^\rho + \Gamma^\alpha_{\nu\rho} V^\mu \Gamma^\rho_{\mu\beta} A^\beta\Big] \ . \end{aligned} \tag{37}$$

Let us now write the above expression as:

$$\begin{aligned}\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\mathbf{A} = \ & V^\nu\Big[\left(\partial_\nu V^\mu\right)\left(\partial_\mu A^\alpha\right) + V^\mu \partial_\nu\partial_\mu A^\alpha + \left(\partial_\nu V^\mu\right)\Gamma^\alpha_{\mu\beta}A^\beta \\ & + V^\mu \Gamma^\alpha_{\mu\beta}\partial_\nu A^\beta + V^\mu \partial_\nu \Gamma^\alpha_{\mu\beta} A^\beta + \Gamma^\alpha_{\nu\rho} V^\mu \partial_\mu A^\rho\Big] \\ & + V^\nu\Big[V^\mu \partial_\nu \Gamma^\alpha_{\mu\beta} A^\beta + \Gamma^\alpha_{\nu\rho} V^\mu \Gamma^\rho_{\mu\beta} A^\beta\Big] \\ = \ & V^\nu\Big[\left(\partial_\nu V^\mu\right)\left(\partial_\mu A^\alpha\right) + V^\mu \partial_\nu\partial_\mu A^\alpha + \left(\partial_\nu V^\mu\right)\Gamma^\alpha_{\mu\beta}A^\beta \\ & + V^\mu \Gamma^\alpha_{\mu\beta}\partial_\nu A^\beta + V^\mu \partial_\nu \Gamma^\alpha_{\mu\beta} A^\beta + \Gamma^\alpha_{\nu\rho} V^\mu \partial_\mu A^\rho\Big] \\ & + V^\nu\Big[V^\mu \partial_\nu \Gamma^\alpha_{\mu\beta} A^\beta + \Gamma^\alpha_{\nu\rho} V^\mu \Gamma^\rho_{\mu\beta} A^\beta\Big] \ , \end{aligned} \tag{38}$$

which may be further simplified as:

$$\begin{aligned}
\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\mathbf{A} = \quad & V^{\nu}\Big[\left(\partial_{\nu}V^{\mu}\right)\left(\partial_{\mu}A^{\alpha} + \Gamma^{\alpha}_{\mu\beta}A^{\beta}\right) + V^{\mu}\partial_{\nu}\partial_{\mu}A^{\alpha} \\
& + V^{\mu}\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta} + V^{\mu}\Gamma^{\alpha}_{\mu\beta}\partial_{\nu}A^{\beta} + V^{\mu}\Gamma^{\alpha}_{\nu\rho}\partial_{\mu}A^{\rho}\Big] \\
& + V^{\nu}\Big[V^{\mu}\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta} + \Gamma^{\alpha}_{\nu\rho}V^{\mu}\Gamma^{\rho}_{\mu\beta}A^{\beta}\Big] \\
= \quad & V^{\nu}\Big[\left(\partial_{\nu}V^{\mu}\right)\left(\nabla_{\mu}A^{\alpha}\right) + V^{\mu}\partial_{\nu}\partial_{\mu}A^{\alpha} \\
& + V^{\mu}\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta} + 2V^{\mu}\Gamma^{\alpha}_{\mu\beta}\partial_{\nu}A^{\beta}\Big] \quad \text{(letting } \rho \to \beta \text{ and } \mu \leftrightarrow \nu\text{)} \\
& + V^{\nu}V^{\mu}\Big[\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta} + \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\beta}A^{\beta}\Big] \ . \qquad (39)
\end{aligned}$$

At this point we remark that all terms in the first square brackets of equation (39) are unchanged under interchange of μ and ν indices, whereas the two terms in the second pair of square brackets are not. As such, if we calculate $2\nabla_{[\nu}\nabla_{\mu]}A^{\alpha}$ we will find that the first set of terms in the square brackets will vanish. Doing this for the second convective derivative we derived we obtain:

$$\begin{aligned}
2\nabla_{[\mathbf{V}}\nabla_{\mathbf{V}]}\mathbf{A} &= V^{\nu}V^{\mu}\left(\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta}A^{\beta} - \partial_{\mu}\Gamma^{\alpha}_{\nu\beta}A^{\beta} - \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta}A^{\beta}\right) \\
&= V^{\nu}V^{\mu}R^{\alpha}{}_{\beta\nu\mu}A^{\beta} \ , \qquad (40)
\end{aligned}$$

thus we obtain an expression which depends on the Riemann curvature tensor. The expression $\nabla_{[\mathbf{V}}\nabla_{\mathbf{V}]}$ (or $\nabla_{[\nu}\nabla_{\mu]}$ in component form) thus measure differences in a vector which is transported in different directions around (say) a closed loop but which reach the same point. This equation is known as the geodesic deviation equation.