General Relativity: Solutions to exercises in Lecture VIII

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Exercise 1

Consider the metric describing, in polar co-ordinates (r, θ), a Euclidean space

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\theta^2 \ . \tag{1}$$

- Calculate the Christoffel symbols and geodesic curves associated with this space, which are given by the geodesic equation
$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0 \ . \tag{2}$$
- Combine the two second-order differential equations describing the geodesic curves into a single first-order differential equation for $r = r(\theta)$.
- What is the differential equation for a straight line in this space?

Solution 1

- First let us consider the components of the metric and their partial derivatives:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \ , \tag{3}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix} \ , \tag{4}$$

$$g_{\mu\nu,r} = \begin{pmatrix} 0 & 0 \\ 0 & 2r \end{pmatrix} \ , \tag{5}$$

$$g_{\mu\nu,\theta} = 0 \ . \tag{6}$$

Next, recall the definition of the Christoffel symbols:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta}\right) \ . \tag{7}$$

Since α can only be r or θ and the metric is diagonal, we may proceed as follows:

$$\begin{aligned}
\Gamma^r_{\beta\gamma} &= \frac{1}{2}g^{rr}\left(\cancel{g_{r\beta,\gamma}} + \cancel{g_{\gamma r,\beta}} - g_{\beta\gamma,r}\right) \\
&= -\frac{1}{2}g^{rr}g_{\beta\gamma,r} \ , \qquad (8) \\
\Gamma^\theta_{\beta\gamma} &= \frac{1}{2}g^{\theta\theta}\left(g_{\theta\beta,\gamma} + g_{\gamma\theta,\beta} - \cancel{g_{\beta\gamma,\theta}}\right) \\
&= \frac{1}{2}g^{\theta\theta}g_{\theta\beta,\gamma} \ . \qquad (9)
\end{aligned}$$

It immediately follows that the only non-zero Christoffel symbols are given by:

$$\Gamma^r_{\theta\theta} = -r \ , \tag{10}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \ . \tag{11}$$

Substituting these expression into the geodesic equation of motion (2) we obtain:

$$\ddot{r} = r\dot{\theta}^2 \ , \tag{12}$$

$$\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} \ , \tag{13}$$

where an overdot denotes differentiation with respect to the affine parameter, λ.

- For the second part of the question we may rewrite equation (13) as:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(r^2\dot{\theta}\right) = 0 \ , \tag{14}$$

which may be integrated to yield

$$\dot{\theta} = \frac{k}{r^2} \ , \tag{15}$$

where k is a constant of integration. Next, starting from the line element and dividing both sides by $\mathrm{d}s^2$ and taking s as affine we may write

$$\dot{r}^2 + r^2\dot{\theta}^2 = 1 \ . \tag{16}$$

Using the chain rule we may write equation (16) as:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 = 1 \ . \tag{17}$$

Substituting equation (15) into equation (17) we obtain:

$$\left[r'(\theta)^2 + r^2\right]\frac{k^2}{r^4} = 1 \ , \tag{18}$$

where a primed quantity denotes differentiation with respect to θ. This may be simplified to yield

$$r'(\theta) = \pm r\sqrt{\frac{r^2}{k^4} - 1} \ . \tag{19}$$

- For the final part of the question, let us integrate equation (19), which describes geodesics in this spacetime. Rearranging both sides of equation (19) gives:

$$\frac{\mathrm{d}r}{r\sqrt{\frac{r^2}{k^4}-1}} = \pm \mathrm{d}\theta \ . \tag{20}$$

Integrating both sides of the above equation then yields:

$$\arctan\left(\sqrt{\frac{r^2}{k^4}-1}\right) = \pm\left(\theta+\theta_0\right) \ . \tag{21}$$

Making use of the identity $\cos\left[\arctan\left(f(x)\right)\right] = \left[1+f(x)^2\right]^{-1/2}$ we may take the cosine of both sides of the above equation, yielding:

$$\sqrt{\frac{k^4}{r^2}} = \cos(\theta+\theta_0) \ , \tag{22}$$

which may be finally written as

$$r\cos(\theta+\theta_0) = k^2 \ , \tag{23}$$

which is precisely the equation of a straight line in polar co-ordinates. Thus the geodesic equations of motion, which we derived in the first part of the question, are straight lines.

Exercise 2

Consider the metric describing the two-dimensional spacetime covered by co-ordinates (t, x) and with metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 - \mathrm{d}t^2}{t^2} \ . \tag{24}$$

- Compute the Christoffel symbols.
- Compute the geodesic curves of this spacetime.

Solution 2

- As in question 1, let us first start by writing down the metric components and their partial derivatives. First consider the components of the metric and their partial derivatives:

$$g_{\mu\nu} = \begin{pmatrix} -t^{-2} & 0 \\ 0 & t^{-2} \end{pmatrix} \ , \tag{25}$$

$$g^{\mu\nu} = \begin{pmatrix} -t^{2} & 0 \\ 0 & t^{2} \end{pmatrix} \ , \tag{26}$$

$$g_{\mu\nu,t} = \begin{pmatrix} 2t^{-3} & 0 \\ 0 & -2t^{-3} \end{pmatrix} \ , \tag{27}$$

$$g_{\mu\nu,x} = 0 \ . \tag{28}$$

Next, recall the definition of the Christoffel symbols:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta} \right) \ . \tag{29}$$

We may now write:

$$\Gamma^{t}_{\beta\gamma} = \frac{1}{2} g^{tt} \left(g_{t\beta,\gamma} + g_{\gamma t,\beta} - g_{\beta\gamma,t} \right) \ , \tag{30}$$

$$\Gamma^{x}_{\beta\gamma} = \frac{1}{2} g^{xx} \left(g_{x\beta,\gamma} + g_{\gamma x,\beta} - \cancel{g_{\beta\gamma,x}} \right) \ . \tag{31}$$

For equation (30) only $\beta = \gamma = t$ or x yields non-zero terms, and for equation (31) only $\beta = x$, $\gamma = t$ (or vice-versa) result in non-vanishing terms. It immediately follows that the only non-zero Christoffel symbols are all identical and are given by:

$$\Gamma^{t}_{tt} = \Gamma^{t}_{xx} = \Gamma^{x}_{tx} = -\frac{1}{t} \ . \tag{32}$$

- For the second part of the question let us first write the geodesic equations of motion for this spacetime. As in exercise 1, an overdot denotes differentiation with respect to the affine parameter. With the Christoffel symbol components we may write the geodesic equations of motion as:

$$\ddot{t} = \frac{1}{t} \left(\dot{t}^2 + \dot{x}^2 \right) \ , \tag{33}$$

$$\ddot{x} = \frac{2}{t} \dot{t} \dot{x} \ , \tag{34}$$

and from the line element we may write

$$\dot{x}^2 - \dot{t}^2 = t^2 \ . \tag{35}$$

We may write equation (34) as

$$\frac{\ddot{x}}{\dot{x}} = 2 \frac{\dot{t}}{t} \ , \tag{36}$$

which may be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\ln \dot{x} \right) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\ln t^2 \right) \ . \tag{37}$$

Integrating both sides of this equation then yields

$$\dot{x} = k t^2 \ , \tag{38}$$

where k is a constant of integration. Substituting equation (38) into equation (35) yields

$$k^2 t^4 - \dot{t}^2 = t^2 \ , \tag{39}$$

which may be solved for $\dot{t}$ to yield

$$\dot{t} = \pm t \sqrt{k^2 t^2 - 1} \ . \tag{40}$$

We may now obtain a differential equation for x as a function of t by dividing equation (38) by equation (40), yielding

$$x'(t) = \pm \frac{k t}{\sqrt{k^2 t^2 - 1}} \ , \tag{41}$$

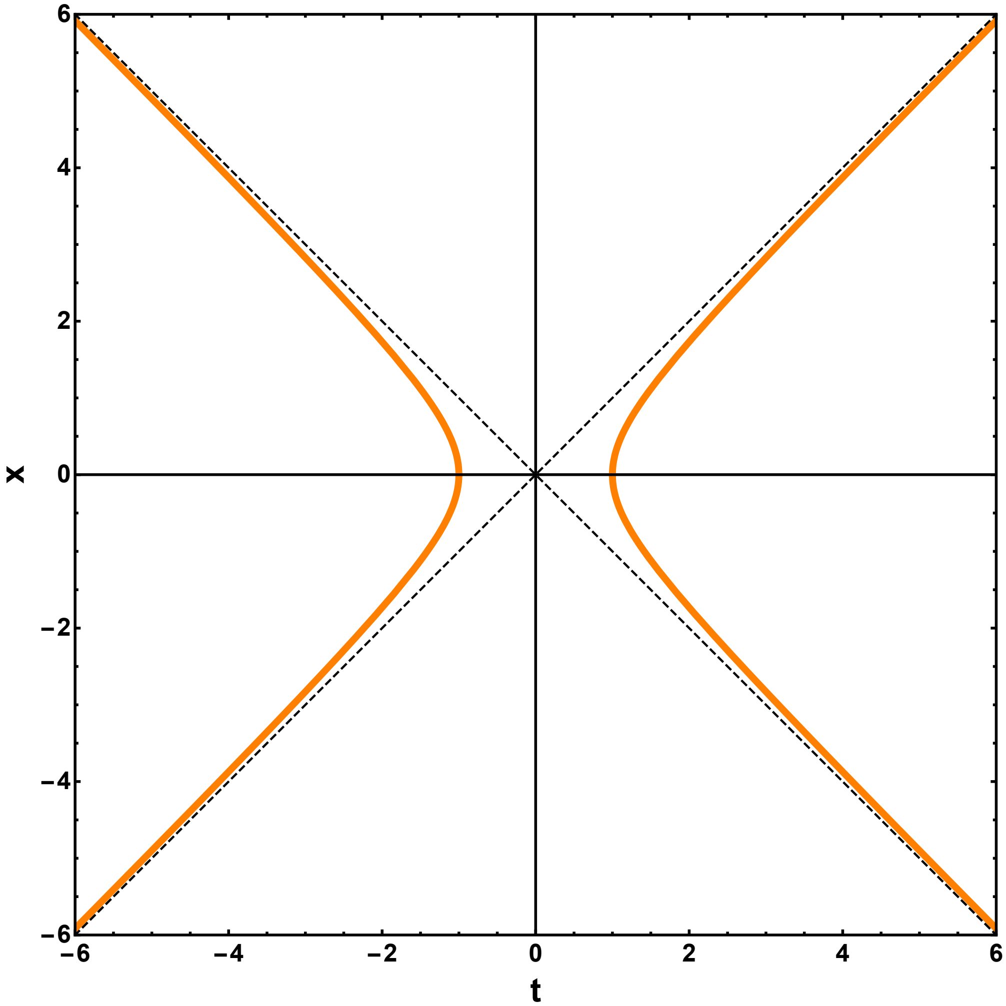

Figure 1: Hyperbolic geodesics as described by equation (43) for the case $x_0 = 0$ and $k = 1$. Note that the geodesics (orange curves) are asymptotic to the lightcone (dashed black line).

which may be integrated to give

$$\begin{aligned} x - x_0 &= \pm \frac{1}{k}\sqrt{k^2 t^2 - 1} \\ &= \pm\sqrt{t^2 - k^{-2}} \ , \end{aligned} \tag{42}$$

where x_0 is a constant of integration. Finally, upon squaring both sides and rearranging we obtain

$$\frac{t^2}{(1/k)^2} - \frac{(x - x_0)^2}{(1/k)^2} = 1 \ , \tag{43}$$

which is the equation of a hyperbola. Thus the geodesics curves in this spacetime are described by hyperbolas. This is illustrated in Figure 1.

Exercise 3

Given a scalar function $\phi \equiv \phi\left(x^{\mu}\right)$, prove the following identity in a co-ordinate basis:

$$\Box\phi := \nabla^{\mu}\nabla_{\mu}\phi = \frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}\,g^{\mu\nu}\partial_{\nu}\phi\right) \ . \tag{44}$$

Solution 3

First, consider the following

$$\begin{aligned} \nabla^{\mu}\nabla_{\mu}\phi &= \left(g^{\mu\nu}\phi_{;\nu}\right)_{;\mu} \\ &= \left(g^{\mu\nu}\phi_{,\nu}\right)_{;\mu} \ , \end{aligned} \tag{45}$$

since ϕ is a scalar quantity. Recall the identity we derived in Problem Sheet 7, question 3, part 5:

$$A^{\mu}{}_{;\mu} = \frac{1}{|g|^{1/2}}\left(|g|^{1/2}A^{\mu}\right)_{,\mu} \ . \tag{46}$$

Using this identity and substituting $A^{\mu} = g^{\mu\nu}\phi_{,\nu}$, we may now write

$$\nabla^{\mu}\nabla_{\mu}\phi = \frac{1}{|g|^{1/2}}\left(|g|^{1/2}g^{\mu\nu}\phi_{,\nu}\right)_{,\mu} \ , \tag{47}$$

which is the desired result, as required. Note that we use $|g|^{1/2}$ and $\sqrt{-g}$ interchangeably.

Exercise 4

Optional: Derive the geodesic equation from the definition of a curve of extremal length.

Solution 4

The Euler-Lagrange equations of motion are derived by extremising the length of a curve. For a given metric tensor $g_{\mu\nu}$ the Lagrangian may be written as

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \ , \tag{48}$$

where, as before, an overdot denotes differentiation with respect to the affine parameter. The Euler-Lagrange equations are by definition written as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^{\alpha}}\right) = \frac{\partial\mathcal{L}}{\partial x^{\alpha}} \ . \tag{49}$$

Let us now derive each term. First we calculate the RHS of (49):

$$\frac{\partial\mathcal{L}}{\partial x^{\alpha}} = \frac{1}{2}g_{\mu\nu,\alpha}\dot{x}^{\mu}\dot{x}^{\nu} \ . \tag{50}$$

For the LHS of equation (49) first consider:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} &= \frac{1}{2} g_{\mu\nu} \left(\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha} \right) \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \left(\frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha} \right) \\
&= \frac{1}{2} g_{\mu\nu} \delta^\mu_\alpha \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \delta^\mu_\alpha \dot{x}^\mu \\
&= \frac{1}{2} g_{\alpha\nu} \dot{x}^\nu + \frac{1}{2} g_{\mu\alpha} \dot{x}^\mu \\
&= g_{\alpha\mu} \dot{x}^\mu \ ,
\end{aligned} \tag{51}$$

where in the last step with have made use of the fact that μ and ν are dummy indices, as well as the metric tensor being symmetric. Now we differentiate with respect to the affine parameter, yielding:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \right) &= \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(g_{\alpha\mu} \right) \dot{x}^\mu + g_{\alpha\mu} \ddot{x}^\mu \\
&= \dot{x}^\beta \frac{\partial}{\partial x^\beta} \left(g_{\alpha\mu} \right) \dot{x}^\mu + g_{\alpha\mu} \ddot{x}^\mu \\
&= g_{\alpha\mu,\beta} \dot{x}^\beta \dot{x}^\mu + g_{\alpha\mu} \ddot{x}^\mu \ .
\end{aligned} \tag{52}$$

Note that the dummy indices β and μ in the first term in equation (52) enable us to expand this term as follows:

$$g_{\alpha\mu,\beta} \dot{x}^\beta \dot{x}^\mu = \frac{1}{2} \left(g_{\alpha\mu,\beta} + g_{\alpha\beta,\mu} \right) \dot{x}^\beta \dot{x}^\mu \ . \tag{53}$$

Using equation (53) we may write the Euler-Lagrange equations as:

$$g_{\alpha\mu} \ddot{x}^\mu + \frac{1}{2} \left(g_{\alpha\mu,\beta} + g_{\alpha\beta,\mu} \right) \dot{x}^\beta \dot{x}^\mu = \frac{1}{2} g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu \ . \tag{54}$$

Bringing all terms to the LHS and relabelling the dummy indices μ and ν in the RHS of equation (54) as β and μ respectively, we obtain

$$g_{\alpha\mu} \ddot{x}^\mu + \frac{1}{2} \left(g_{\alpha\mu,\beta} + g_{\alpha\beta,\mu} - g_{\beta\mu,\alpha} \right) \dot{x}^\beta \dot{x}^\mu = 0. \tag{55}$$

Next, multiply both sides of this expression by $g^{\delta\alpha}$, yielding

$$\ddot{x}^\delta + \frac{1}{2} g^{\delta\alpha} \left(g_{\alpha\mu,\beta} + g_{\alpha\beta,\mu} - g_{\beta\mu,\alpha} \right) \dot{x}^\beta \dot{x}^\mu = 0 \ , \tag{56}$$

where we have used the fact that $g^{\delta\alpha} g_{\alpha\mu} = \delta^\delta_\mu$. Recalling the definition of the Christoffel symbols this expression may be written more succinctly as

$$\ddot{x}^\delta + \Gamma^\delta_{\mu\beta} \dot{x}^\mu \dot{x}^\beta = 0 \ . \tag{57}$$

Let us now relabel the dummy indices as $\delta \to \alpha$, $\mu \to \beta$ and $\beta \to \gamma$, enabling us to rewrite (57) in the more familiar form

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0 \ , \tag{58}$$

which is precisely the geodesic equation, as required.