General Relativity: Solutions to exercises in Lecture VII

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Exercise 1

Show that if $\mathbf{g}$ is the metric tensor, then its covariant derivative is zero, i.e.

$$\nabla_\lambda g_{\mu\nu} = 0 \ . \tag{1}$$

Solution 1

By definition $\boldsymbol{\nabla} A_\mu$ is a vector. As such we may write

$$\nabla_\lambda A_\mu = g_{\mu\nu} \left(\nabla_\lambda A^\nu \right) \ . \tag{2}$$

We may also write

$$\begin{aligned} \nabla_\lambda A_\mu &= \nabla_\lambda \left(g_{\mu\nu} A^\nu \right) \\ &= \left(\nabla_\lambda g_{\mu\nu} \right) A^\nu + g_{\mu\nu} \left(\nabla_\lambda A^\nu \right) \ . \end{aligned} \tag{3}$$

Using equation (2) we may rewrite the above expression as

$$\nabla_\lambda A_\mu = \left(\nabla_\lambda g_{\mu\nu} \right) A^\nu + \nabla_\lambda A_\mu \ , \tag{4}$$

from which it immediately follows that

$$\nabla_\lambda g_{\mu\nu} = 0 \ , \tag{5}$$

as required.

Exercise 2

Using the results of exercise 1, drive the following definition of the Christoffel symbols

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma} \right) \ . \tag{6}$$

Solution 2

Consider the following three expressions for the covariant derivative of the (covariant) metric tensor

$$\nabla_\lambda g_{\mu\nu} = g_{\mu\nu,\lambda} - \Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} - \Gamma^\alpha_{\lambda\nu} g_{\mu\alpha} \quad (=0) \ , \tag{7}$$
$$\nabla_\mu g_{\nu\lambda} = g_{\nu\lambda,\mu} - \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda} g_{\nu\alpha} \quad (=0) \ , \tag{8}$$
$$\nabla_\nu g_{\lambda\mu} = g_{\lambda\mu,\nu} - \Gamma^\alpha_{\nu\lambda} g_{\alpha\mu} - \Gamma^\alpha_{\nu\mu} g_{\lambda\alpha} \quad (=0) \ , \tag{9}$$

where we have (evenly) permuted the covariant indices, as well as having made use of the result of exercise 1, namely that $\nabla_\lambda g_{\mu\nu} = 0$. We have also written partial derivatives as subscript commas for the sake of brevity.

To prove the result, subtract the last two expressions from the first, i.e. (7) $-$[(8) + (9)]. This yields

$$g_{\mu\nu,\lambda} - g_{\nu\lambda,\mu} - g_{\lambda\mu,\nu} - \boldsymbol{\Gamma^\alpha_{\lambda\mu} g_{\alpha\nu}} - \boldsymbol{\Gamma^\alpha_{\lambda\nu} g_{\mu\alpha}} + \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} + \boldsymbol{\Gamma^\alpha_{\mu\lambda} g_{\nu\alpha}} + \boldsymbol{\Gamma^\alpha_{\nu\lambda} g_{\alpha\mu}} + \Gamma^\alpha_{\nu\mu} g_{\lambda\alpha} = 0 \ . \tag{10}$$

By the torsion-free condition (i.e. $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$) and symmetry of the metric tensor (i.e. $g_{\mu\nu} = g_{\nu\mu}$) the red and blue terms cancel, yielding

$$g_{\mu\nu,\lambda} - g_{\nu\lambda,\mu} - g_{\lambda\mu,\nu} + 2\,\Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} = 0 \ , \tag{11}$$

which upon re-arranging gives

$$g_{\alpha\lambda}\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\left(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right) \ . \tag{12}$$

Multiplying both sides by $g^{\beta\lambda}$ gives

$$\delta^\beta_\alpha \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\beta\lambda}\left(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right) \ , \tag{13}$$

which immediately simplifies to

$$\Gamma^\beta_{\mu\nu} = \frac{1}{2} g^{\beta\lambda}\left(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right) \ . \tag{14}$$

Finally, making the substitutions $\beta \to \alpha$, $\mu \to \beta$, $\nu \to \gamma$ and $\lambda \to \delta$ we obtain the result

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}\right) \ , \tag{15}$$

as required.

Exercise 3

Prove the following identities:

$$\partial_\gamma g_{\alpha\beta} = \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} \ , \tag{16}$$
$$g_{\alpha\mu}\partial_\gamma g^{\mu\beta} = -g^{\mu\beta}\partial_\gamma g_{\alpha\mu} \ , \tag{17}$$
$$\partial_\gamma g^{\alpha\beta} = -\Gamma^\alpha_{\mu\gamma} g^{\mu\beta} - \Gamma^\beta_{\mu\gamma} g^{\mu\alpha} \ , \tag{18}$$
$$(\ln|g|)_{,\alpha} = g^{\mu\nu} g_{\mu\nu,\alpha} \ , \tag{19}$$
$$\nabla_\mu A^\mu = \frac{1}{|g|^{1/2}}\partial_\mu\left(|g|^{1/2}A^\mu\right) \quad \text{in a coordinate basis.} \tag{20}$$

Solution 3

- For the first part consider the action of the covariant derivative on $g_{\alpha\beta}$:

$$\nabla_\gamma g_{\alpha\beta} = g_{\alpha\beta,\gamma} - \Gamma^\lambda_{\gamma\alpha} g_{\lambda\beta} - \Gamma^\lambda_{\gamma\beta} g_{\alpha\lambda} = 0 \ . \tag{21}$$

  Rearranging yields

$$\begin{aligned} g_{\alpha\beta,\gamma} &= \Gamma^\lambda_{\gamma\alpha} g_{\lambda\beta} + \Gamma^\lambda_{\gamma\beta} g_{\alpha\lambda} \\ &= g_{\lambda\beta}\Gamma^\lambda_{\alpha\gamma} + g_{\alpha\lambda}\Gamma^\lambda_{\beta\gamma} \\ &= \Gamma_{\beta\alpha\gamma} + \Gamma_{\alpha\beta\gamma} \ , \end{aligned} \tag{22}$$

  as required.

- For the second part consider the following expression:

$$\left(g_{\alpha\mu} g^{\mu\beta}\right)_{,\gamma} = \left(\delta^\beta_\alpha\right)_{,\gamma} = 0 \ , \tag{23}$$

  which may also be expanded as

$$\begin{aligned} \left(g_{\alpha\mu} g^{\mu\beta}\right)_{,\gamma} &= g_{\alpha\mu,\gamma} g^{\mu\beta} + g_{\alpha\mu} g^{\mu\beta}{}_{,\gamma} \\ &= 0 \ . \end{aligned} \tag{24}$$

  Rearranging the above expression, and writing partial derivatives explicitly, we obtain

$$g_{\alpha\mu}\partial_\gamma g^{\mu\beta} = -g^{\mu\beta}\partial_\gamma g_{\alpha\mu} \ , \tag{25}$$

  as required.

- For the third part, let us consider the action of the covariant derivative on the contravariant metric tensor:

$$\nabla_\gamma g^{\alpha\beta} = g^{\alpha\beta}{}_{,\gamma} + \Gamma^\alpha_{\gamma\mu} g^{\mu\beta} + \Gamma^\beta_{\gamma\mu} g^{\alpha\mu} = 0 \ . \tag{26}$$

  Rearranging and making use of the symmetry conditions of the metric tensor and Christoffel symbol yields

$$\partial_\gamma g^{\alpha\beta} = -\Gamma^\alpha_{\mu\gamma} g^{\mu\beta} - \Gamma^\beta_{\mu\gamma} g^{\mu\alpha} \ , \tag{27}$$

  as required.

- For the fourth part consider the metric tensor $g_{\alpha\beta}$, which is a rank-2 tensor and specifically a matrix. For matrices one may consider the Jacobi matrix formula:

$$\frac{\partial}{\partial x^\alpha}\det\left[g_{\mu\nu}(x^\alpha)\right] = \mathrm{Tr}\left[\mathrm{adj}\left(g_{\alpha\beta}(x^\alpha)\right)\frac{\partial g_{\mu\nu}(x^\alpha)}{\partial x^\alpha}\right] \ , \tag{28}$$

  where the adjugate of a matrix may be written as

$$\begin{aligned} \mathrm{adj}\left(g_{\alpha\beta}\right) &= \det\left(g_{\alpha\beta}\right)\left(g_{\alpha\beta}\right)^{-1} \\ &= g^{\alpha\beta}\det\left(g_{\alpha\beta}\right) \\ &= g^{\alpha\beta}|g| \ , \end{aligned} \tag{29}$$

where we have omitted writing the dependence of the metric on co-ordinates for brevity, and written the determinant of the metric tensor as $|g|$.

We may now rewrite the Jacobi identity in equation (28) in logarithmic form as

$$\begin{aligned} \frac{\partial}{\partial x^\alpha}\left[\ln\det\left(g_{\alpha\beta}\right)\right] &= \frac{1}{\det\left(g_{\alpha\beta}\right)}\frac{\partial}{\partial x^\alpha}\left[\det\left(g_{\alpha\beta}\right)\right] \\ &= \frac{1}{|g|}\mathrm{Tr}\left[g^{\alpha\beta}|g|g_{\mu\nu,\alpha}\right] \\ &= \mathrm{Tr}\left[g^{\alpha\beta}g_{\mu\nu,\alpha}\right] \\ &= g^{\mu\nu}g_{\mu\nu,\alpha}\ . \end{aligned} \tag{30}$$

This may be written more succinctly as

$$\left(\ln|g|\right)_{,\alpha} = g^{\mu\nu}g_{\mu\nu,\alpha}\ , \tag{31}$$

as required.

- For the fifth and final part, consider the action of the covariant derivative on A^μ:

$$\nabla_\mu A^\mu = A^\mu{}_{,\mu} + \Gamma^\mu_{\mu\nu}A^\nu\ . \tag{32}$$

Using the definition of the covariant derivative derived in exercise 2, we may write $\Gamma^\mu_{\mu\nu}$ as:

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2}g^{\mu\delta}\left(g_{\delta\mu,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta}\right)\ . \tag{33}$$

Whilst it is not immediately obvious, it can be shown that the last two terms in brackets in equation (33) vanish. Consider the following:

$$\begin{aligned} g^{\mu\delta}\left(g_{\nu\delta,\mu} - g_{\mu\nu,\delta}\right) &= g^{\mu\delta}\partial_\mu g_{\nu\delta} - g^{\mu\delta}\partial_\delta g_{\mu\nu} \\ &= \partial^\delta g_{\nu\delta} - \partial^\mu g_{\mu\nu} \\ &= \partial^\delta g_{\delta\nu} - \partial^\mu g_{\mu\nu} \quad (g_{\nu\delta} = g_{\delta\nu}) \\ &= \partial^\mu g_{\mu\nu} - \partial^\mu g_{\mu\nu} \quad (\delta \text{ is a dummy index}) \\ &= 0\ . \end{aligned} \tag{34}$$

Consequently we may rewrite equation (33) as

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2}g^{\mu\delta}g_{\delta\mu,\nu}\ . \tag{35}$$

In the fourth part of this exercise we showed that $\left(\ln|g|\right)_{,\nu} = g^{\mu\delta}g_{\mu\delta,\nu}$. Using this we may write

$$\begin{aligned} \Gamma^\mu_{\mu\nu} &= \frac{1}{2}g^{\mu\delta}g_{\delta\mu,\nu} \\ &= \frac{1}{2}\left(\ln|g|\right)_{,\nu} \\ &= \left(\ln|g|^{1/2}\right)_{,\nu} \\ &= \frac{\left(|g|^{1/2}\right)_{,\nu}}{|g|^{1/2}} \end{aligned} \tag{36}$$

Returning to equation (32) we may now re-write the expression as

$$\begin{aligned}
\nabla_\mu A^\mu &= A^\mu{}_{,\mu} + \frac{\left(|g|^{1/2}\right)_{,\nu}}{|g|^{1/2}} A^\nu \\
&= A^\mu{}_{,\mu} + \frac{\left(|g|^{1/2}\right)_{,\mu}}{|g|^{1/2}} A^\mu \quad \text{(relabel dummy index)} \\
&= \frac{1}{|g|^{1/2}} \left[|g|^{1/2} A^\mu{}_{,\mu} + \left(|g|^{1/2}\right)_{,\mu} A^\mu \right] \\
&= \frac{1}{|g|^{1/2}} \left(|g|^{1/2} A^\mu \right)_{,\mu} \\
&\equiv \frac{1}{|g|^{1/2}} \partial_\mu \left(|g|^{1/2} A^\mu \right) \ ,
\end{aligned} \tag{37}$$

as required.

Exercise 4

Optional: The covariant derivative of a contravariant vector U^μ is

$$\nabla_\nu U^\mu := \partial_\nu U^\mu + \Gamma^\mu_{\nu\lambda} U^\lambda \ . \tag{38}$$

Use this expression to obtain the covariant derivative of the covariant vector U_μ.

Solution 4

There are several ways one can go about proving this. Let us consider two such methods.

- **Method 1**

  Consider the following:

$$\begin{aligned}
\nabla_\nu \left(V^\mu U_\mu \right) &= V^\mu_{;\nu} U_\mu + V^\mu U_{\mu;\nu} \\
&= V^\mu_{,\nu} U_\mu + \Gamma^\mu_{\nu\lambda} V^\lambda U_\mu + V^\mu U_{\mu;\nu} \ ,
\end{aligned} \tag{39}$$

  where the subscript $_{;\nu}$ denotes the covariant differentiation with respect to x^ν and we have used the definition of $V^\mu_{;\nu}$. Since the quantity $V^\mu U_\mu$ is a scalar we may also write

$$\begin{aligned}
\nabla_\nu \left(V^\mu U_\mu \right) &= \partial_\nu \left(V^\mu U_\mu \right) \\
&= V^\mu_{,\nu} U_\mu + V^\mu U_{\mu,\nu} \ .
\end{aligned} \tag{40}$$

  Combining the above two equations yields

$$V^\mu_{,\nu} U_\mu + V^\mu U_{\mu,\nu} = V^\mu_{,\nu} U_\mu + \Gamma^\mu_{\nu\lambda} V^\lambda U_\mu + V^\mu U_{\mu;\nu} \ , \tag{41}$$

  which simplifies to

$$V^\mu U_{\mu,\nu} = \Gamma^\mu_{\nu\lambda} V^\lambda U_\mu + V^\mu U_{\mu;\nu} \ , \tag{42}$$

from which we may obtain

$$V^{\mu}U_{\mu;\nu} = V^{\mu}U_{\mu,\nu} - \Gamma^{\mu}_{\nu\lambda}V^{\lambda}U_{\mu} \ . \tag{43}$$

Now let us set $V^{\mu} = \delta^{\mu}_{\beta}$, which gives

$$\delta^{\mu}_{\beta}U_{\mu;\nu} = \delta^{\mu}_{\beta}U_{\mu,\nu} - \Gamma^{\mu}_{\nu\lambda}\delta^{\lambda}_{\beta}U_{\mu} \ , \tag{44}$$

which simplifies to

$$U_{\beta;\nu} = U_{\beta,\nu} - \Gamma^{\mu}_{\nu\beta}U_{\mu} \ , \tag{45}$$

where upon setting $\mu \leftrightarrow \alpha$ and then $\beta \rightarrow \mu$ we obtain

$$U_{\mu;\nu} = U_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu}U_{\alpha} \ , \tag{46}$$

as required.

- **Method 2**

$$\begin{aligned} \nabla_{\nu}U_{\mu} &= \nabla_{\nu}\left(g_{\mu\alpha}U^{\alpha}\right) \\ &= \cancel{\left(\nabla_{\nu}g_{\mu\alpha}\right)} + g_{\mu\alpha}\nabla_{\nu}U^{\alpha} \\ &= g_{\mu\alpha}\left(U^{\alpha}_{\ ,\nu} + \Gamma^{\alpha}_{\nu\lambda}U^{\lambda}\right) \\ &= g_{\mu\alpha}\left(U^{\alpha}_{\ ,\nu}\right) + g_{\mu\alpha}\Gamma^{\alpha}_{\nu\lambda}U^{\lambda} \ . \end{aligned} \tag{47}$$

Now consider the expression

$$\begin{aligned} \left(g_{\mu\alpha}U^{\alpha}\right)_{,\nu} &= U_{\mu,\nu} \\ &= g_{\mu\alpha,\nu}U^{\alpha} + g_{\mu\alpha}\left(U^{\alpha}_{\ ,\nu}\right) \ , \end{aligned} \tag{48}$$

which upon rearrangement yields

$$g_{\mu\alpha}\left(U^{\alpha}_{\ ,\nu}\right) = U_{\mu,\nu} - g_{\mu\alpha,\nu}U^{\alpha} \ . \tag{49}$$

Substituting equation (49) into equation (47) yields

$$\nabla_{\nu}U_{\mu} = U_{\mu,\nu} - g_{\mu\alpha,\nu}U^{\alpha} + g_{\mu\alpha}\Gamma^{\alpha}_{\nu\lambda}U^{\lambda} \ . \tag{50}$$

From exercise 3, part 1, recall the identity

$$g_{\mu\alpha,\nu} = g_{\lambda\alpha}\Gamma^{\lambda}_{\nu\mu} + g_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha} \ . \tag{51}$$

This enables us to rewrite the last two terms in equation (50) as:

$$\begin{aligned} -g_{\mu\alpha,\nu}U^{\alpha} + g_{\mu\alpha}\Gamma^{\alpha}_{\nu\lambda}U^{\lambda} &= -g_{\lambda\alpha}\Gamma^{\lambda}_{\nu\mu}U^{\alpha} - g_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha}U^{\alpha} + g_{\mu\alpha}\Gamma^{\alpha}_{\nu\lambda}U^{\lambda} \\ &= -g_{\lambda\alpha}\Gamma^{\lambda}_{\nu\mu} - g_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha}U^{\alpha} + g_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha}U^{\alpha} \quad (\alpha \leftrightarrow \lambda \text{ in last term}) \\ &= -g_{\lambda\alpha}\Gamma^{\lambda}_{\nu\mu}u^{\alpha} \\ &= -\Gamma^{\lambda}_{\nu\mu}u_{\lambda} \ . \end{aligned} \tag{52}$$

We may now use the above expression to rewrite equation (50) as

$$\nabla_{\nu}U_{\mu} = U_{\mu,\nu} - \Gamma^{\lambda}_{\nu\mu}U_{\lambda} \ , \tag{53}$$

which may be rewritten as

$$\nabla_{\nu}U_{\mu} := \partial_{\nu}U_{\mu} - \Gamma^{\lambda}_{\nu\mu}U_{\lambda} \ , \tag{54}$$

as required.