General Relativity: Solutions to exercises in Lecture VI

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Exercise 1

Define the antisymmetric tensor $\mathbf{F}$ as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Use the results from the previous exercises to show that

$$F_{\mu\nu} := 2\,\partial_{[\mu} A_{\nu]} \ . \tag{1}$$

Show that such a definition implies that

$$F_{\alpha\beta,\nu} + F_{\beta\nu,\alpha} + F_{\nu\alpha,\beta} = 0 \ . \tag{2}$$

Solution 1

For the first part we may simply write

$$\begin{aligned} F_{\mu\nu} &= 2\left[\frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right] \\ &= 2\,\partial_{[\mu} A_{\nu]} \ . \end{aligned} \tag{3}$$

For the second part of the question we must write out each of the three terms explicitly. For the first term in equation (2) we obtain

$$\begin{aligned} F_{\alpha\beta,\nu} &= \partial_\nu\left(F_{\alpha\beta}\right) \\ &= \partial_\nu\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \\ &= \partial_\nu\partial_\alpha A_\beta - \partial_\nu\partial_\beta A_\alpha \ . \end{aligned} \tag{4}$$

For the second term

$$\begin{aligned} F_{\beta\nu,\alpha} &= \partial_\alpha\left(F_{\beta\nu}\right) \\ &= \partial_\alpha\left(\partial_\beta A_\nu - \partial_\nu A_\beta\right) \\ &= \partial_\alpha\partial_\beta A_\nu - \partial_\alpha\partial_\nu A_\beta \ . \end{aligned} \tag{5}$$

Finally, for the third term

$$\begin{aligned} F_{\nu\alpha,\beta} &= \partial_\beta\left(F_{\nu\alpha}\right) \\ &= \partial_\beta\left(\partial_\nu A_\alpha - \partial_\alpha A_\nu\right) \\ &= \partial_\beta\partial_\nu A_\alpha - \partial_\beta\partial_\alpha A_\nu \ . \end{aligned} \tag{6}$$

Since $\partial_\alpha\partial_\beta\mathbf{F} = \partial_\beta\partial_\alpha\mathbf{F}$, summing equations (4)–(6) leads to cancellation of terms, giving the result in equation (2), as required.

Exercise 2

Consider a vector $\mathbf{V}$ with components V^μ relative to a co-ordinate basis, i.e.

$$\mathbf{V} = V^\mu \partial_\mu = V^\mu \mathbf{e}_\mu \ . \tag{7}$$

Define an object given by the partial derivative of the components of $\mathbf{V}$, i.e.

$$U_\nu{}^\mu := \partial_\nu V^\mu \ . \tag{8}$$

Show that $U_\nu{}^\mu$ is not a tensor. What are the implications of this result? What can be done to construct a tensor out of measuring the derivative of a tensor?

Solution 2

From equation (8) we may write

$$\begin{aligned} U_\nu &= \partial_\nu \mathbf{V} \\ &= \partial_\nu \left(V^\mu \mathbf{e}_\mu \right) \\ &= \partial_\nu V^\mu \mathbf{e}_\mu + V^\mu \partial_\nu \mathbf{e}_\mu \ . \end{aligned} \tag{9}$$

For the second term in the above expression we may think of it as a vector written in terms of some basis vectors. Let us re-write this as $\partial_\nu \mathbf{e}_\mu = \Gamma^\alpha_{\mu\nu} \mathbf{e}_\alpha$. We may now write equation (9) as

$$\begin{aligned} U_\nu &= \partial_\nu V^\mu \mathbf{e}_\mu + V^\mu \Gamma^\alpha_{\mu\nu} \mathbf{e}_\alpha \\ &= \partial_\nu V^\mu \mathbf{e}_\mu + V^\alpha \Gamma^\mu_{\alpha\nu} \mathbf{e}_\mu \quad (\alpha \leftrightarrow \mu \text{ in the second term}) \\ &= \left(\partial_\nu V^\mu + V^\alpha \Gamma^\mu_{\alpha\nu} \right) \mathbf{e}_\mu \\ &= \left(\nabla_\nu V^\mu \right) \mathbf{e}_\mu \ , \end{aligned} \tag{10}$$

and thus we may write

$$U_\nu{}^\mu = \nabla_\nu V^\mu \ , \tag{11}$$

where the ∇_ν we have introduced is defined as the *covariant derivative.*

Consider the term $\partial_\nu \mathbf{e}_\mu = \Gamma^\alpha_{\mu\nu} \mathbf{e}_\alpha$. In flat (Minkowski) spacetime, in Cartesian co-ordinates, $\partial_\nu \mathbf{e}_\mu$ must vanish as the $\mathbf{e}_\mu$ are all constant, and thus $\Gamma^\alpha_{\mu\nu}$ must also be zero. However, in the same Minkowski spacetime, transforming to (for example) spherical polar co-ordinates one would find the basis vector components are not constant and are in fact functionally dependent on r and θ. As such, $\partial_\nu \mathbf{e}_\mu$ would be non-zero in Minkowski spacetime. Since a tensor quantity is defined independently of any co-ordinate system, the quantity $U_\nu{}^\mu$ cannot be a tensor.

The partial derivative is not a good differential operator when spacetime is not Euclidean but by construction the covariant derivative does define the components of a tensor.

Exercise 3

Consider a line element in three-dimensional space

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\phi^2 \ , \tag{12}$$

with a co-ordinate basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$.

- Construct the corresponding orthonormal basis $\{\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\theta}}, \mathbf{e}_{\hat{\phi}}\}$
- Compute the structure coefficients $C^{\theta}_{\hat{r}\hat{\theta}}$ and $C^{\theta}_{r\theta}$. What is the difference between the two?
- Compute the structure coefficients $C^{\theta}_{\hat{r}\hat{\phi}}$, $C^{\phi}_{\hat{r}\hat{\phi}}$, $C^{\theta}_{\hat{\theta}\hat{\phi}}$ and $C^{\phi}_{\hat{\theta}\hat{\phi}}$. Are there others that are non-zero?

Solution 3

Since our metric is diagonal we can immediately read of the orthonormal basis vector components as

$$\mathbf{e}_{\hat{r}} = \mathbf{e}_r \ , \qquad \mathbf{e}_{\hat{\theta}} = \frac{1}{r}\mathbf{e}_\theta \ , \qquad \mathbf{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\mathbf{e}_\phi \ , \tag{13}$$

from which it is straightforward to show that $\mathbf{e}_{\hat{r}} \cdot \mathbf{e}_{\hat{r}} = 1$, $\mathbf{e}_{\hat{\theta}} \cdot \mathbf{e}_{\hat{\theta}} = 1$ and $\mathbf{e}_{\hat{\phi}} \cdot \mathbf{e}_{\hat{\phi}} = 1$. To convince ourselves this is correct, consider the transformation between the co-ordinate basis and orthonormal basis

$$\mathrm{d}x^{\hat{i}} = \Lambda^{\hat{i}}{}_j \, \mathrm{d}x^j \ , \tag{14}$$

where

$$\Lambda^{\hat{i}}{}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{pmatrix} \ . \tag{15}$$

Using the co-ordinate transformation we may show that

$$\begin{aligned} \mathrm{d}\hat{r} &= \mathrm{d}x^{\hat{1}} \\ &= \Lambda^{\hat{1}}{}_j \mathrm{d}x^j \\ &= \Lambda^{\hat{1}}{}_1 \mathrm{d}x^1 \\ &= \mathrm{d}r \ . \end{aligned} \tag{16}$$

Similarly, one may show that

$$\mathrm{d}\hat{\theta} = r\,\mathrm{d}\theta \ , \tag{17}$$

$$\mathrm{d}\hat{\phi} = r\sin\theta\,\mathrm{d}\phi \ . \tag{18}$$

Now let us write the line element in terms of the orthonormal basis components and prove equivalence

$$\begin{aligned} \mathrm{d}s^2 &= g_{\hat{r}\hat{r}}\mathrm{d}\hat{r}^2 + g_{\hat{\theta}\hat{\theta}}\mathrm{d}\hat{\theta}^2 + g_{\hat{\phi}\hat{\phi}}\mathrm{d}\hat{\phi}^2 \\ &= \mathrm{d}\hat{r}^2 + \mathrm{d}\hat{\theta}^2 + \mathrm{d}\hat{\phi}^2 \\ &= \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \ , \end{aligned} \tag{19}$$

hence the orthonormal basis vector components are correct.

For the next part of the question recall the definition of the Lie brackets of any two basis vectors, which may be written in terms of the same basis as

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = C^{\gamma}_{\alpha\beta}\mathbf{e}_\gamma \ , \tag{20}$$

where the components $C^{\gamma}_{\alpha\beta}$ are termed the structure coefficients. By definition a set of basis vectors with all of its structure coefficients vanishing is a co-ordinate basis. We may now write

$$\begin{aligned} C^{\gamma}_{\alpha\beta} &= [\mathbf{e}_\alpha, \mathbf{e}_\beta]^\gamma \\ &= \mathbf{e}_\alpha{}^\nu \partial_\nu \, \mathbf{e}_\beta{}^\gamma - \mathbf{e}_\beta{}^\nu \partial_\nu \, \mathbf{e}_\alpha{}^\gamma \; . \end{aligned} \tag{21}$$

For the first structure coefficient, applying the above machinery we find

$$\begin{aligned} C^{\theta}_{\hat{r}\hat{\theta}} &= [\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\theta}}]^\theta \\ &= \mathbf{e}_{\hat{r}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{\theta}}{}^\theta - \mathbf{e}_{\hat{\theta}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{r}}{}^\theta \\ &= \mathbf{e}_{\hat{r}}{}^r \partial_r \left(\frac{1}{r}\right) - \mathbf{e}_{\hat{\theta}}{}^\theta \partial_\theta \, \cancel{\mathbf{e}_{\hat{r}}{}^\theta} \\ &= -\frac{1}{r^2} \; . \end{aligned} \tag{22}$$

As mentioned previously, $C^{\theta}_{r\theta} = 0$ since $\{\mathbf{e}_i\}$ is a co-ordinate basis. For the final four requested structure components we apply the same procedure for calculation. The results are as follows

$$\begin{aligned} C^{\theta}_{\hat{r}\hat{\phi}} &= \left[\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\phi}}\right]^\theta \\ &= \mathbf{e}_{\hat{r}}{}^\nu \partial_\nu \, \cancel{\mathbf{e}_{\hat{\phi}}{}^\theta} - \mathbf{e}_{\hat{\phi}}{}^\nu \partial_\nu \, \cancel{\mathbf{e}_{\hat{r}}{}^\theta} \\ &= 0 \; , \end{aligned} \tag{23}$$

$$\begin{aligned} C^{\phi}_{\hat{r}\hat{\phi}} &= \left[\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\phi}}\right]^\phi \\ &= \mathbf{e}_{\hat{r}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{\phi}}{}^\phi - \mathbf{e}_{\hat{\phi}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{r}}{}^\phi \\ &= \mathbf{e}_{\hat{r}}{}^r \partial_r \, \mathbf{e}_{\hat{\phi}}{}^\phi - \mathbf{e}_{\hat{\phi}}{}^\nu \partial_\nu \, \cancel{\mathbf{e}_{\hat{r}}{}^\phi} \\ &= \partial_r \left(\frac{1}{r\sin\theta}\right) \\ &= -\frac{1}{r^2\sin\theta} \; , \end{aligned} \tag{24}$$

$$\begin{aligned} C^{\theta}_{\hat{\theta}\hat{\phi}} &= \left[\mathbf{e}_{\hat{\theta}}, \mathbf{e}_{\hat{\phi}}\right]^\theta \\ &= \mathbf{e}_{\hat{\theta}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{\phi}}{}^\theta - \mathbf{e}_{\hat{\phi}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{\theta}}{}^\theta \\ &= \mathbf{e}_{\hat{\theta}}{}^\nu \partial_\nu \, \cancel{\mathbf{e}_{\hat{\phi}}{}^\theta} - \mathbf{e}_{\hat{\phi}}{}^\phi \partial_\phi \, \mathbf{e}_{\hat{\theta}}{}^\theta \\ &= -\frac{1}{r\sin\theta}\partial_\phi\left(r\right) \\ &= 0 \; , \end{aligned} \tag{25}$$

and

$$
\begin{aligned}
C^{\phi}_{\hat{\theta}\hat{\phi}} &= \left[\mathbf{e}_{\hat{\theta}}, \mathbf{e}_{\hat{\phi}}\right]^{\phi} \\
&= \mathbf{e}_{\hat{\theta}}{}^{\nu}\partial_{\nu}\,\mathbf{e}_{\hat{\phi}}{}^{\phi} - \mathbf{e}_{\hat{\phi}}{}^{\nu}\partial_{\nu}\,\mathbf{e}_{\hat{\theta}}{}^{\phi} \\
&= \mathbf{e}_{\hat{\theta}}{}^{\theta}\partial_{\theta}\,\mathbf{e}_{\hat{\phi}}{}^{\phi} - \mathbf{e}_{\hat{\phi}}{}^{\nu}\partial_{\nu}\,\cancel{\mathbf{e}_{\hat{\theta}}{}^{\phi}} \\
&= \frac{1}{r}\partial_{\theta}\left(\frac{1}{r\sin\theta}\right) \\
&= -\frac{\cos\theta}{r^2\sin^2\theta}\ .
\end{aligned} \tag{26}
$$

There are no other non-zero structure coefficients.