General Relativity: Solutions to exercises in Lecture IV

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Exercise 1

Using a co-ordinate system (t, r, θ, ϕ), consider the metric line element given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \ , \tag{1}$$

where $\kappa = -1,\ 0,\ 1$.

- Show that a new co-ordinate system (t, χ, θ, ϕ) the line element (1) can be rewritten as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\chi^2 + f(\chi)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \ . \tag{2}$$

- Find the form of the function $f(\chi)$ for $\kappa = -1$, 0 and 1.
- Discuss the properties of the metric in the case of $\kappa = 0$. [Hint: two metrics $\mathbf{g}$ and $\mathbf{g}'$ are conformally related if it is possible to express them as $\mathbf{g} = \Omega\,\mathbf{g}'$, where $\Omega \equiv \Omega(x^\mu)$ is a generic function and is referred to as the *conformal factor*].

Solution 1

From the invariance of the line element we may write

$$\begin{aligned}\mathrm{d}s^2 &= -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \\ &= -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\chi^2 + f(\chi)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \\ &= \mathrm{d}s'^2 \ .\end{aligned}$$

Letting $\mathrm{d}t = \mathrm{d}\theta = \mathrm{d}\phi = 0$ we obtain the relation

$$\frac{\mathrm{d}r}{\sqrt{1-\kappa r^2}} = \mathrm{d}\chi \ . \tag{3}$$

This expression may be integrated directly to yield χ as a function of r, yielding:

$$\chi = \begin{cases} \operatorname{arcsinh} r + c \ , \\ r + c \ , \\ \arcsin r + c \ , \end{cases} \tag{4}$$

for $\kappa = -1$, 0 and 1 respectively, and where c is a constant of integration. Note the inverse hyperbolic function identity $\operatorname{arcsinh} r = \ln|r + \sqrt{1+r^2}|$, which is also a solution for $\kappa = -1$. Since $f(\chi) = r$ we obtain the result

$$f(\chi) = \begin{cases} \sinh \chi \ , \\ \chi \ , \\ \sin \chi \ , \end{cases} \tag{5}$$

for $\kappa = -1$, 0 and 1 respectively, and where we have assumed $c = 0$.

For the final part of this question, setting $\kappa = 0$ yields the line element as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \ . \tag{6}$$

By factoring out the expansion factor $a(t)$ we obtain

$$\mathrm{d}s^2 = a^2(t)\left[-\frac{\mathrm{d}t^2}{a^2(t)} + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \ . \tag{7}$$

Let us define the "conformal time" $\tilde{t}$, where $\mathrm{d}\tilde{t}^2 = \mathrm{d}t^2/a(t)^2$. We may now re-write the metric (7) as

$$\begin{aligned} \mathrm{d}s^2 &= a^2(t)\left[-\mathrm{d}\tilde{t}^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \\ &= a^2(t)\,\mathrm{d}s^2_{\text{Minkowski}} \ , \end{aligned} \tag{8}$$

where the Minkowski line element is the line element for flat space. Thus in the case $\kappa = 0$ the metric is conformally flat. This metric is in general known as the (Friedmann-Lemaitre) Robertson-Walker (FL)RW metric and is widely used in cosmology to describe an expanding universe.

Exercise 2

Using a co-ordinate system $(\eta, \chi, \theta, \phi)$, consider the metric line element given by

$$\mathrm{d}s^2 = \Omega^2\left[-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \ . \tag{9}$$

Consider now a new co-ordinate system $(\tau, \rho, \theta, \phi)$ where

$$\tau = \frac{2\sin\eta}{\cos\chi + \cos\eta} \tag{10}$$

$$\rho = \frac{2\sin\chi}{\cos\chi + \cos\eta} \ , \tag{11}$$

and find the expression of the metric (9) in this new co-ordinate system. Discuss the properties of this new metric.

Solution 2

To calculate the expression for the new metric we must use the following co-ordinate transformation:

$$g^{\alpha'\beta'} = \Lambda^{\alpha'}_{\ \alpha} \Lambda^{\beta'}_{\ \beta} g^{\alpha\beta} \ . \tag{12}$$

We must calculate the contravariant metric components since we only have (τ, ρ) in terms of (η, χ) and not the inverse relationship. It is much simpler to calculate the contravaraint metric components and then calculate the matrix inverse of the contravariant metric than to define the inverse transformation. Since our metric is diagonal we may exploit the fact that $g^{\alpha\beta} = 1/g_{\alpha\beta}$. Evaluating the non-zero components of the transformation matrix, we obtain:

$$\begin{aligned} \Lambda^{0'}_{\ 0} &= \frac{\partial \tau}{\partial \eta} \\ &= \frac{2(1 + \cos\eta \cos\chi)}{(\cos\chi + \cos\eta)^2} \\ &= \frac{\partial \rho}{\partial \chi} \\ &= \Lambda^{1'}_{\ 1} \ , \end{aligned} \tag{13}$$

$$\begin{aligned} \Lambda^{1'}_{\ 0} &= \frac{\partial \rho}{\partial \eta} \\ &= \frac{2\sin\chi \sin\eta}{(\cos\chi + \cos\eta)^2} \\ &= \frac{\partial \tau}{\partial \chi} \\ &= \Lambda^{0'}_{\ 1} \ , \end{aligned} \tag{14}$$

and

$$\Lambda^{2'}_{\ 2} = \Lambda^{3'}_{\ 3} = 1 \ . \tag{15}$$

Note that the transformation matrix is diagonal. The contravariant metric may now be calculated coefficient by coefficient, yielding

$$\begin{aligned} g^{0'0'} &= \left(\Lambda^{0'}_{\ 0}\right)^2 g^{00} + \left(\Lambda^{0'}_{\ 1}\right)^2 g^{11} \\ &= -\frac{1}{\Omega^2}\left[\left(\Lambda^{0'}_{\ 0}\right)^2 - \left(\Lambda^{0'}_{\ 1}\right)^2\right] \\ &= -\frac{1}{\Omega^2}\frac{4}{(\cos\chi + \cos\eta)^2} \ , \end{aligned} \tag{16}$$

$$\begin{aligned} g^{1'1'} &= \left(\Lambda^{1'}_{\ 0}\right)^2 g^{00} + \left(\Lambda^{1'}_{\ 1}\right)^2 g^{11} \\ &= \frac{1}{\Omega^2}\left[\left(\Lambda^{1'}_{\ 1}\right)^2 - \left(\Lambda^{1'}_{\ 0}\right)^2\right] \\ &= \frac{1}{\Omega^2}\left[\left(\Lambda^{0'}_{\ 0}\right)^2 - \left(\Lambda^{0'}_{\ 1}\right)^2\right] \\ &= -g^{0'0'} \ , \end{aligned} \tag{17}$$

$$
\begin{aligned}
g^{2'2'} &= \frac{1}{\Omega^2 \sin^2 \chi} \\
&= g^{22} \\
&= \frac{4}{\Omega^2 \left(\cos \chi + \cos \eta\right)^2 \rho^2} \\
&= -\frac{1}{\rho^2} g^{0'0'} \ ,
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
g^{3'3'} &= \frac{1}{\Omega^2 \sin^2 \theta \sin^2 \chi} \\
&= g^{33} \\
&= \frac{1}{\sin^2 \theta} g^{2'2'} \\
&= -\frac{1}{\rho^2 \sin^2 \theta} g^{0'0'} \ .
\end{aligned}
\tag{19}
$$

Thus we may write the contravariant metric components in the new co-ordinate system as

$$
g^{\alpha'\beta'} = \frac{1}{\Omega^2} \frac{4}{\left(\cos \chi + \cos \eta\right)^2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \left(\rho^2\right)^{-1} & 0 \\ 0 & 0 & 0 & \left(\rho^2 \sin^2 \theta\right)^{-1} \end{pmatrix} \ . \tag{20}
$$

Let us now define a new conformal factor

$$
\widetilde{\Omega}^2 \equiv \frac{\Omega^2 \left(\cos \chi + \cos \eta\right)^2}{4} \ , \tag{21}
$$

which immediately enables us to write the covariant components of our metric tensor in the new co-ordinate system as

$$
g_{\alpha'\beta'} = \widetilde{\Omega}^2 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ 0 & 0 & 0 & \rho^2 \sin^2 \theta \end{pmatrix} \ . \tag{22}
$$

We may now write our metric in the new co-ordinate system as follows

$$
\mathrm{d}s^2 = \widetilde{\Omega}^2 \left(-\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2 + \rho^2 \sin^2 \theta \, \mathrm{d}\phi^2\right) \ . \tag{23}
$$

As can be seen in the above expression, the new metric is conformally flat.

Exercise 3

Given the four-vector $\mathbf{u}$ such that $u^\alpha u_\alpha = -1$ and the tensor $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$, prove the following identities

$$h_{\mu\nu}u^\mu = 0 \ , \qquad h^\mu{}_\nu h^\lambda{}_\mu = h^\lambda{}_\nu \ , \qquad h^\mu{}_\mu = 3 \ . \tag{24}$$

Solution 3

For the first identity, consider the following

$$\begin{aligned} h_{\mu\nu}u^\mu &= g_{\mu\nu}u^\mu + u_\mu u_\nu u^\mu \\ &= u_\nu + u_\nu \left(u_\mu u^\mu\right) \\ &= u_\nu - u_\nu \\ &= 0 \ . \end{aligned} \tag{25}$$

For the second identity, we must first derive an expression for $h^\mu{}_\nu$ as follows

$$\begin{aligned} h^\mu{}_\nu &= g^{\mu\alpha}h_{\alpha\nu} \\ &= \delta^\mu{}_\nu + u^\mu u_\nu \ . \end{aligned} \tag{26}$$

Using this we may write the following

$$\begin{aligned} h^\mu{}_\nu h^\lambda{}_\mu &= \left(\delta^\mu{}_\nu + u^\mu u_\nu\right)\left(\delta^\lambda{}_\mu + u^\lambda u_\mu\right) \\ &= \delta^\mu{}_\nu \delta^\lambda{}_\mu + \delta^\mu{}_\nu u^\lambda u_\mu + \delta^\lambda{}_\mu u^\mu u_\nu + u^\mu u_\nu u^\lambda u_\mu \\ &= \delta^\lambda_\nu + u^\lambda u_\nu + u^\lambda u_\nu + u^\lambda u_\nu \left(u^\mu u_\mu\right) \\ &= \delta^\lambda_\nu + u^\lambda u_\nu \\ &= h^\lambda{}_\nu \ . \end{aligned} \tag{27}$$

For the third and final identity, consider the following

$$\begin{aligned} h^\mu{}_\mu &= g^{\mu\nu}h_{\nu\mu} \\ &= g^{\mu\nu}g_{\mu\nu} + g^{\mu\nu}u_\mu u_\nu \\ &= \delta^\mu{}_\mu + u^\mu u_\mu \\ &= 4 - 1 \\ &= 3 \ . \end{aligned} \tag{28}$$

Note: the tensor $h_{\mu\nu}$ defines a projection onto a hypersurface orthogonal to u^μ (i.e. $h_{\mu\nu}u^\mu u^\nu = 0$). For any non-null vector u^μ (i.e. $u^\mu u_\mu \neq 0$), one may define the projection operator orthogonal to u^μ as

$$\begin{aligned} \mathrm{P}_{\underline{u}} &\equiv h_{\mu\nu} \\ &= g_{\mu\nu} - \frac{u_\mu u_\nu}{u_\mu u^\mu} \ . \end{aligned} \tag{29}$$

Exercise 4

Consider the following antisymmetric tensor

$$F_{\alpha\beta} = -2E_{[\alpha}u_{\beta]} + \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma \, u_\delta \ . \tag{30}$$

Express the vectors **E** and **H** in terms of the tensor **F**. [Hint: contract $F_{\alpha\beta}$ with u^β & $\epsilon^{\alpha\beta\gamma\delta}$ respectively.]

Solution 4

First, let us write the expression for the antisymmetric part of $E_\alpha u_\beta$ out in full, which reads as

$$E_{[\alpha}u_{\beta]} = \frac{1}{2}\left(E_\alpha u_\beta - E_\beta u_\alpha\right) \ . \tag{31}$$

We may then substitute this expression into equation (30), yielding

$$F_{\alpha\beta} = -E_\alpha u_\beta + E_\beta u_\alpha + \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma \, u_\delta \ . \tag{32}$$

Let us work with the above expression for the remainder of the question. Contracting $F_{\alpha\beta}$ with u^β yields

$$\begin{aligned} F_{\alpha\beta}u^\beta &= -E_\alpha u_\beta u^\beta + E_\beta u_\alpha u^\beta + \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma \, u_\delta u^\beta \\ &= E_\alpha + E_\beta u_\alpha u^\beta + \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma \, u_\delta u^\beta \ . \end{aligned} \tag{33}$$

Before proceeding further, let us turn our attention to the last term in equation (33). The Levi-Civita tensor may be re-written in a fully contravariant form as

$$\epsilon_{\alpha\beta}{}^{\gamma\delta} = g_{\alpha\mu}\, g_{\beta\nu}\, \epsilon^{\mu\nu\gamma\delta} \ , \tag{34}$$

which simplifies the third term in equation (33) as follows

$$\begin{aligned} \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma \, u_\delta u^\beta &= g_{\alpha\mu}\, g_{\beta\nu}\, \epsilon^{\mu\nu\gamma\delta} u_\delta u^\beta \, H_\gamma \\ &= g_{\alpha\mu}\, \epsilon^{\mu\nu\gamma\delta} u_\delta u_\nu \, H_\gamma \qquad \text{(lower index with } g_{\beta\nu}\text{)} \\ &= g_{\alpha\mu}\, \epsilon^{\mu\delta\gamma\nu} u_\nu u_\delta \, H_\gamma \qquad (\delta \leftrightarrow \nu \text{ as dummy indices}) \\ &= -g_{\alpha\mu}\, \epsilon^{\mu\nu\gamma\delta} u_\nu u_\delta \, H_\gamma \qquad (\text{permute } \delta \leftrightarrow \nu \text{ in } \epsilon^{\mu\delta\gamma\nu}) \\ &= -g_{\alpha\mu}\, \epsilon^{\mu\nu\gamma\delta} u_\delta u_\nu \, H_\gamma \qquad \text{(compare with second line)} \\ &= 0 \ . \end{aligned} \tag{35}$$

We thus obtain

$$\begin{aligned} F_{\alpha\beta}u^\beta &= E_\alpha + E_\beta u_\alpha u^\beta \\ &= h^\beta{}_\alpha \, E_\beta \ , \end{aligned} \tag{36}$$

as required. This may also be written as

$$F_{\alpha\beta}u^\beta = h_{\alpha\beta}\, E^\beta \ . \tag{37}$$

For the second part of the question, first recall the definition of the dual of a tensor

$$F^{*\gamma\delta} = \frac{1}{2} F_{\alpha\beta}\, \epsilon^{\alpha\beta\gamma\delta} \ . \tag{38}$$

Contracting $F_{\alpha\beta}$ with $\epsilon^{\alpha\beta\gamma\delta}$ yields

$$\begin{aligned} F_{\alpha\beta}\epsilon^{\alpha\beta\gamma\delta} &= 2F^{*\gamma\delta} \\ &= -E_\alpha u_\beta\, \epsilon^{\alpha\beta\gamma\delta} + E_\beta u_\alpha\, \epsilon^{\alpha\beta\gamma\delta} + \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma\, u_\delta\, \epsilon^{\alpha\beta\gamma\delta} \ . \end{aligned} \tag{39}$$

Let us attack the third term in the above expression by employing the identity we derived in equation (34) as

$$\epsilon_{\alpha\beta}{}^{\gamma\delta} = g^{\mu\gamma} g^{\nu\delta} \epsilon_{\alpha\beta\mu\nu} \ . \tag{40}$$

With this expression we may re-write the third term as

$$\begin{aligned} \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma\, u_\delta\, \epsilon^{\alpha\beta\gamma\delta} &= g^{\mu\gamma} g^{\nu\delta}\, H_\gamma\, u_\delta\, \epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\gamma\delta} \\ &= H^\mu\, u^\nu\, \epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\gamma\delta} \ . \end{aligned} \tag{41}$$

We may then expand the contraction over the Levi-Civita tensors as

$$\begin{aligned} \epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\gamma\delta} &= -2!\, \delta^{\gamma\delta}_{\mu\nu} \\ &= -2 \begin{vmatrix} \delta^\gamma_\mu & \delta^\gamma_\nu \\ \delta^\delta_\mu & \delta^\delta_\nu \end{vmatrix} \\ &= 2\left(\delta^\gamma_\nu\, \delta^\delta_\mu - \delta^\gamma_\mu\, \delta^\delta_\nu\right) \ , \end{aligned} \tag{42}$$

from which we may immediately simplify equation (41), yielding

$$\begin{aligned} H^\mu\, u^\nu\, \epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\gamma\delta} &= 2H^\mu\, u^\nu\, \left(\delta^\gamma_\nu\, \delta^\delta_\mu - \delta^\gamma_\mu\, \delta^\delta_\nu\right) \\ &= 2\left(H^\delta u^\gamma - H^\gamma u^\delta\right) \end{aligned} \tag{43}$$

We may now re-write equation (39) as

$$2F^{*\gamma\delta} = -E_\alpha u_\beta\, \epsilon^{\alpha\beta\gamma\delta} + E_\beta u_\alpha\, \epsilon^{\alpha\beta\gamma\delta} + 2\left(H^\delta u^\gamma - H^\gamma u^\delta\right) \ . \tag{44}$$

Recall from equation (35) the vanishing of the contraction of the Levi-Civita tensor over two indices with two 4-vectors. This suggests to us that contracting equation (44) with u_δ will allow us to eliminate the first two terms in (44). With this knowledge in mind, contracting with u_δ yields

$$\begin{aligned} 2F^{*\gamma\delta}u_\delta &= -E_\alpha u_\beta u_\delta\, \epsilon^{\alpha\beta\gamma\delta} + E_\beta u_\alpha u_\delta\, \epsilon^{\alpha\beta\gamma\delta} + 2\left(H^\delta u^\gamma u_\delta - H^\gamma u^\delta u_\delta\right) \\ &= 2\left(H^\delta u^\gamma u_\delta - H^\gamma u^\delta u_\delta\right) \ , \end{aligned} \tag{45}$$

from which it immediately follows that

$$\begin{aligned} F^{*\gamma\delta}u_\delta &= H^\delta u^\gamma u_\delta - H^\gamma u^\delta u_\delta \\ &= H^\gamma + H^\delta u^\gamma u_\delta \ . \end{aligned} \tag{46}$$

As before, the above expressions may be written more succinctly in terms of the projection tensor as

$$
\begin{aligned}
F^{*\gamma\delta}u_\delta &= h^\gamma{}_\delta H^\delta \\
&= h^{\gamma\delta} H_\delta \ . \qquad (47)
\end{aligned}
$$

For a physical interpretation consider an orthonormal comoving frame with $u^\mu = (1,0,0,0)$ and $u_\mu = (-1,0,0,0)$, i.e. $u^\mu u_\mu = -1$. In this frame

$$
\begin{aligned}
F_{\alpha\beta}u^\beta &= F_{\alpha 0} \\
&= E_\alpha + E_0 u_\alpha \ . \qquad (48)
\end{aligned}
$$

When $\alpha = 0$

$$
\begin{aligned}
F_{00} &= E_0 - E_0 \\
&= 0 \ . \qquad (49)
\end{aligned}
$$

Additionally

$$
\begin{aligned}
F_{i0} &= E_i + E_0 u_i \\
&= E_i \ , \qquad (50)
\end{aligned}
$$

where $i = 1,2,3$. If $F_{\alpha\beta}$ is the electromagnetic field tensor then E_i is the 3-vector of the electric field. Next consider the dual tensor

$$
\begin{aligned}
F^{*\gamma\delta}u_\delta &= -F^{*\gamma 0} \\
&= H^\gamma - H^0 u^\gamma \ . \qquad (51)
\end{aligned}
$$

When $\gamma = 0$ then

$$
\begin{aligned}
F^{*00} &= -\left(H^0 - H^0\right) \\
&= 0 \ . \qquad (52)
\end{aligned}
$$

Additionally

$$
\begin{aligned}
F^{*i0} &= -(H^i - H^0 u^i) \\
&= -H^i \ , \qquad (53)
\end{aligned}
$$

where again $i = 1,2,3$. H^i can be interpreted as the 3-vector of the magnetic field.