General Relativity: Solutions to exercises in Lecture XIV

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Exercise 1

Using the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \ , \tag{1}$$

where $\kappa = -1,\, 0,\, 1$. Compute:

- the non-zero Christoffel symbols
- the non-zero components of the Ricci tensor
- the expression for the Ricci scalar

Solution 1

- The first part of the question asks us to calculate the non-zero Christoffel symbol components of the FLRW metric. Let us begin by writing the Lagrangian for the FLRW metric:

$$\mathcal{L} = \frac{1}{2}\left(-t'^2 + \frac{a^2}{1-\kappa r^2}\,r'^2 + a^2 r^2\,\theta'^2 + a^2 r^2\sin^2\theta\,\phi'^2\right) \ , \tag{2}$$

where primed quantities ($'$) denote differentiation with respect to the affine parameter, λ. We have also written $a \equiv a(t)$ for brevity. Next we employ the Euler-Lagrange equations, which may be written as:

$$\frac{\partial\mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial x'^\alpha}\right) = 0 \ . \tag{3}$$

The Euler-Lagrange equations are equivalent to the geodesic equations of motion (see Lecture XIII, exercise 2) and so we can read off the Christoffel symbol components directly. First, we consider the t-component:

$$\frac{\partial\mathcal{L}}{\partial t} = a\,\dot{a}\left(\frac{r'^2}{1-\kappa r^2} + r^2\,\theta'^2 + r^2\sin^2\theta\,\phi'^2\right) \ , \tag{4}$$

$$\frac{\partial\mathcal{L}}{\partial t'} = -t' \ , \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial t'}\right) = -t'' \ , \tag{6}$$

where an overdot ($\dot{}$) denotes differentiation with respect to t. We immediately obtain:

$$t'' = -\frac{a\,\dot{a}}{1-\kappa r^2}\,r'^2 - a\,\dot{a}\left(r^2\,\theta'^2 + r^2\sin^2\theta\,\phi'^2\right)\ . \tag{7}$$

We may now read off the Christoffel symbol components directly, obtaining:

$$\begin{aligned}
{\Gamma^t}_{rr} &= \frac{a\dot{a}}{1-\kappa r^2}\ , && (8)\\
{\Gamma^t}_{\theta\theta} &= a\,\dot{a}\,r^2\ , && (9)\\
{\Gamma^t}_{\phi\phi} &= a\,\dot{a}\,r^2\sin^2\theta\ . && (10)
\end{aligned}$$

Next, we consider the r-component of the Euler-Lagrange equations:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial r} &= a^2 r\left[\frac{\kappa}{(1-\kappa r^2)^2} + \theta'^2 + \sin^2\theta\,\phi'^2\right]\ , && (11)\\
\frac{\partial\mathcal{L}}{\partial r'} &= \frac{a^2}{1-\kappa r^2}\,r'\ , && (12)\\
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial r'}\right) &= \frac{2a\,\dot{a}}{1-\kappa r^2}t'\,r' + \frac{\kappa r}{(1-\kappa r^2)^2}r'^2 + \frac{a^2}{1-\kappa r^2}r''\ . && (13)
\end{aligned}$$

We thus obtain the geodesic equation of motion as:

$$r'' = -2\frac{\dot{a}}{a}t'\,r' - \frac{\kappa r}{1-\kappa r^2}r'^2 + r(1-\kappa r^2)\theta'^2 + r^2\sin^2\theta(1-\kappa r^2\phi'^2)\ , \tag{14}$$

from which the Christoffel symbols are immediately obtained as:

$$\begin{aligned}
{\Gamma^r}_{rt} = {\Gamma^r}_{tr} &= \frac{\dot{a}}{a}\ , && (15)\\
{\Gamma^r}_{rr} &= \frac{\kappa r}{1-\kappa r^2}\ , && (16)\\
{\Gamma^r}_{\theta\theta} &= -r(1-\kappa r^2)\ , && (17)\\
{\Gamma^r}_{\phi\phi} &= -r^2\sin^2\theta(1-\kappa r^2)\ . && (18)
\end{aligned}$$

We now consider the θ-component of the Euler-Lagrange equations:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta} &= a^2 r^2\sin\theta\cos\theta\,\phi'^2\ , && (19)\\
\frac{\partial\mathcal{L}}{\partial\theta'} &= a^2 r^2\,\theta'\ , && (20)\\
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\theta'}\right) &= 2\,a\,\dot{a}\,r^2 t'\,\theta' + 2a^2 r\,r'\,\theta' + a^2 r^2\,\theta''\ . && (21)
\end{aligned}$$

This gives the geodesic equation of motion for θ as:

$$\theta'' = -2\frac{\dot{a}}{a}\,t'\,\theta' - \frac{2}{r}\,r'\,\theta' + \sin\theta\cos\theta\,\phi'^2\ , \tag{22}$$

from which the Christoffel symbols immediately follow as:

$$\Gamma^{\theta}{}_{t\theta} = \Gamma^{\theta}{}_{\theta t} = \frac{\dot{a}}{a} , \tag{23}$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r} , \tag{24}$$

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta . \tag{25}$$

Finally, we consider the ϕ-component of the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 , \tag{26}$$

$$\frac{\partial \mathcal{L}}{\partial \phi'} = a^2 r^2 \sin^2\theta\, \phi'\, \theta' , \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) = 2a\,\dot{a}\,r^2 \sin^2\theta\, t'\, \phi' + 2a^2 r\, \sin^2\theta\, r'\, \phi' + 2a^2 r^2 \sin\theta\cos\theta\, \theta'\, \phi' + a^2 r^2 \sin^2\theta\, \phi'' , \tag{28}$$

from which we obtain the geodesic equation of motion for ϕ as:

$$\phi'' = -2\frac{\dot{a}}{a}\, t'\, \phi' - \frac{2}{r}\, r'\, \phi' - 2\cot\theta\, \theta'\, \phi' . \tag{29}$$

Thus the final non-zero Christoffel symbols read:

$$\Gamma^{\phi}{}_{t\phi} = \Gamma^{\phi}{}_{\phi t} = \frac{\dot{a}}{a} , \tag{30}$$

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r} , \tag{31}$$

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta . \tag{32}$$

- For the second part of the question, recall the definition of the Ricci tensor, which is the contraction of the Riemann curvature tensor over the first and third indices. This may be written as:

$$\begin{aligned} R_{\mu\nu} &= R^{\alpha}{}_{\mu\alpha\nu} \\ &= \Gamma^{\alpha}{}_{\mu\nu,\alpha} + \Gamma^{\rho}{}_{\mu\nu}\Gamma^{\alpha}{}_{\rho\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} - \Gamma^{\rho}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\rho\nu} . \end{aligned} \tag{33}$$

The FLRW metric is spherically symmetric and possesses no off-diagonal terms, i.e. $g_{\mu\nu} = 0$ if $\mu \neq \nu$. Let us consider the four terms in the definition of the Ricci tensor for $\mu \neq \nu$.

In this case $\Gamma^{\alpha}{}_{\mu\nu,\alpha} = 0\,\forall\,\mu \neq \nu$ (see the Christoffel symbol components). The third term $\Gamma^{\alpha}{}_{\mu\alpha,\nu} = 0$ also, since $\Gamma^{\alpha}{}_{\mu\alpha} \propto f(x^{\mu})$ and thus $\Gamma^{\alpha}{}_{\mu\alpha,\nu} = 0$ since $\mu \neq \nu$.

For a spherically symmetric metric $\Gamma^{\alpha}{}_{\mu\nu} = 0$ if $\alpha \neq \mu \neq \nu$, which follows from the definition of the Christoffel symbols. Using this, it may be shown that for all six independent combinations of (μ, ν), with $\mu \neq \nu$, that $\Gamma^{\rho}{}_{\mu\nu}\Gamma^{\alpha}{}_{\rho\alpha} - \Gamma^{\rho}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\rho\nu} = 0$. We may thus conclude that $R_{\mu\nu} = 0$ for $\mu \neq \nu$.

Let us consider the diagonal components of the Ricci tensor term-by-term. First the tt component:

$$R_{tt} = \cancelto{0}{\Gamma^{\alpha}{}_{tt,\alpha}} + \cancelto{0}{\Gamma^{\rho}{}_{tt}}\Gamma^{\alpha}{}_{\rho\alpha} - \Gamma^{\alpha}{}_{t\alpha,t} - \Gamma^{\rho}{}_{t\alpha}\Gamma^{\alpha}{}_{\rho t} . \tag{34}$$

For the third and fourth terms we obtain:

$$\begin{aligned}
\Gamma^{\alpha}{}_{t\alpha,t} &= \Gamma^{i}{}_{ti,t} \\
&= 3\left(\frac{a\,\ddot{a}-\dot{a}^2}{a^2}\right)\ , \qquad (35)\\
\Gamma^{\rho}{}_{t\alpha}\Gamma^{\alpha}{}_{\rho t} &= \Gamma^{\rho}{}_{ti}\Gamma^{i}{}_{\rho t} \\
&= \left(\Gamma^{i}{}_{ti}\right)^2 \ \text{(only } \rho = i \text{ gives a nonzero result)} \\
&= 3\frac{\dot{a}^2}{a^2}\ , \qquad (36)
\end{aligned}$$

where the index i denotes spatial co-ordinates (r, θ, ϕ). Thus we immediately find:

$$R_{tt} = -3\frac{\ddot{a}}{a}\ . \qquad (37)$$

We next consider the rr component of the Ricci tensor:

$$R_{rr} = \Gamma^{\alpha}{}_{rr,\alpha} + \Gamma^{\rho}{}_{rr}\Gamma^{\alpha}{}_{\rho\alpha} - \Gamma^{\alpha}{}_{r\alpha,r} - \Gamma^{\rho}{}_{r\alpha}\Gamma^{\alpha}{}_{\rho r}\ . \qquad (38)$$

For the first term:

$$\begin{aligned}
\Gamma^{\alpha}{}_{rr,\alpha} &= \Gamma^{t}{}_{rr,t} + \Gamma^{r}{}_{rr,r} \\
&= \frac{\dot{a}^2 + a\,\ddot{a} + \kappa}{1-\kappa r^2} + \frac{2\kappa^2 r^2}{\left(1-\kappa r^2\right)^2}\ . \qquad (39)
\end{aligned}$$

For the second term:

$$\begin{aligned}
\Gamma^{\rho}{}_{rr}\Gamma^{\alpha}{}_{\rho\alpha} &= \Gamma^{t}{}_{rr}\Gamma^{\alpha}{}_{t\alpha} \\
&= \Gamma^{t}{}_{rr}\left(\Gamma^{i}{}_{ti}\right) + \Gamma^{r}{}_{rr}\left(\Gamma^{i}{}_{ri}\right) \\
&= \frac{3\dot{a}^2 + 2\kappa}{1-\kappa r^2} + \frac{\kappa^2 r^2}{\left(1-\kappa r^2\right)^2}\ . \qquad (40)
\end{aligned}$$

For the third term:

$$\begin{aligned}
\Gamma^{\alpha}{}_{r\alpha,r} &= \Gamma^{i}{}_{ri,r} \\
&= \frac{2\kappa^2 r^2}{\left(1-\kappa r^2\right)^2} + \frac{\kappa}{1-\kappa r^2} - \frac{2}{r^2}\ . \qquad (41)
\end{aligned}$$

For the fourth term:

$$\begin{aligned}
\Gamma^{\rho}{}_{r\alpha}\Gamma^{\alpha}{}_{\rho r} &= \Gamma^{\rho}{}_{rt}\Gamma^{t}{}_{\rho r} + \Gamma^{\rho}{}_{ri}\Gamma^{i}{}_{\rho r} \\
&= \Gamma^{r}{}_{rt}\Gamma^{t}{}_{rr} + \Gamma^{t}{}_{ri}\Gamma^{i}{}_{tr} + \Gamma^{i}{}_{ri}\Gamma^{i}{}_{ir} \\
&= 2\Gamma^{r}{}_{rt}\Gamma^{t}{}_{rr} + \left(\Gamma^{i}{}_{ri}\right)^2 \\
&= \frac{\kappa^2 r^2}{\left(1-\kappa r^2\right)^2} + \frac{2\dot{a}^2}{1-\kappa r^2} + \frac{2}{r^2}\ . \qquad (42)
\end{aligned}$$

We thus obtain:

$$R_{rr} = \frac{2\kappa + 2\dot{a}^2 + a\,\ddot{a}}{1-\kappa r^2}\ . \qquad (43)$$

The $\theta\theta$ component of the Ricci tensor yields:

$$R_{\theta\theta} = \Gamma^{\alpha}{}_{\theta\theta,\alpha} + \Gamma^{\rho}{}_{\theta\theta}\Gamma^{\alpha}{}_{\rho\alpha} - \Gamma^{\alpha}{}_{\theta\alpha,\theta} - \Gamma^{\rho}{}_{\theta\alpha}\Gamma^{\alpha}{}_{\rho\theta} \ . \tag{44}$$

For the first term we obtain:

$$\begin{aligned} \Gamma^{\alpha}{}_{\theta\theta,\alpha} &= \Gamma^{t}{}_{\theta\theta,t} + \Gamma^{r}{}_{\theta\theta,r} \\ &= -1 + 3\kappa r^2 + r^2(\dot{a}^2 + a\,\ddot{a}) \ . \end{aligned} \tag{45}$$

For the second term we obtain:

$$\begin{aligned} \Gamma^{\rho}{}_{\theta\theta}\Gamma^{\alpha}{}_{\rho\alpha} &= \Gamma^{t}{}_{\theta\theta}\Gamma^{\alpha}{}_{t\alpha} + \Gamma^{r}{}_{\theta\theta}\Gamma^{\alpha}{}_{r\alpha} \\ &= \Gamma^{t}{}_{\theta\theta}\Gamma^{i}{}_{ti} + \Gamma^{r}{}_{\theta\theta}\Gamma^{i}{}_{ri} \\ &= 3\dot{a}^2 r^2 + \kappa r^2 - 2 \ . \end{aligned} \tag{46}$$

For the third term we obtain:

$$\begin{aligned} \Gamma^{\alpha}{}_{\theta\alpha,\theta} &= \Gamma^{\phi}{}_{\theta\phi,\theta} \\ &= -\mathrm{cosec}^2\theta \ . \end{aligned} \tag{47}$$

For the fourth term we obtain:

$$\begin{aligned} \Gamma^{\rho}{}_{\theta\alpha}\Gamma^{\alpha}{}_{\rho\theta} &= \Gamma^{\rho}{}_{\theta t}\Gamma^{t}{}_{\rho\theta} + \Gamma^{\rho}{}_{\theta i}\Gamma^{i}{}_{\rho\theta} \\ &= \Gamma^{\rho}{}_{\theta t}\Gamma^{t}{}_{\rho\theta} + \Gamma^{t}{}_{\theta i}\Gamma^{i}{}_{t\theta} + \Gamma^{j}{}_{\theta i}\Gamma^{i}{}_{j\theta} \\ &= \Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + \Gamma^{t}{}_{\theta i}\Gamma^{i}{}_{t\theta} + \Gamma^{j}{}_{\theta i}\Gamma^{i}{}_{j\theta} \\ &= \Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + \Gamma^{t}{}_{\theta\theta}\Gamma^{\theta}{}_{t\theta} + \Gamma^{j}{}_{\theta i}\Gamma^{i}{}_{j\theta} \\ &= 2\Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + \Gamma^{j}{}_{\theta i}\Gamma^{i}{}_{j\theta} \\ &= 2\Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + \Gamma^{r}{}_{\theta\theta}\Gamma^{\theta}{}_{r\theta} + \Gamma^{\theta}{}_{\theta r}\Gamma^{r}{}_{\theta\theta} + \Gamma^{\phi}{}_{\theta\phi}\Gamma^{\phi}{}_{\phi\theta} \\ &= 2\Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + 2\Gamma^{r}{}_{\theta\theta}\Gamma^{\theta}{}_{r\theta} + \left(\Gamma^{\phi}{}_{\theta\phi}\right)^2 \\ &= 2\dot{a}^2 r^2 + 2\kappa r^2 - 2 + \cot^2\theta \ . \end{aligned} \tag{48}$$

We thus obtain:

$$R_{\theta\theta} = r^2\left(2\kappa + 2\dot{a}^2 + a\,\ddot{a}\right) \ . \tag{49}$$

Finally, we consider the $\phi\phi$ component of the Ricci tensor:

$$R_{\phi\phi} = \Gamma^{\alpha}{}_{\phi\phi,\alpha} + \Gamma^{\rho}{}_{\phi\phi}\Gamma^{\alpha}{}_{\rho\alpha} - \cancelto{0}{\Gamma^{\alpha}{}_{\phi\alpha,\phi}} - \Gamma^{\rho}{}_{\phi\alpha}\Gamma^{\alpha}{}_{\rho\phi} \ . \tag{50}$$

The first term gives:

$$\begin{aligned} \Gamma^{\alpha}{}_{\phi\phi,\alpha} &= \Gamma^{i}{}_{\phi\phi,i} \\ &= -\cos^2\theta + \left(\dot{a}^2 + a\,\ddot{a} + 3\kappa\right) r^2 \sin^2\theta \ . \end{aligned} \tag{51}$$

The second term gives:

$$\begin{aligned} \Gamma^{\rho}{}_{\phi\phi}\Gamma^{\alpha}{}_{\rho\alpha} &= \Gamma^{t}{}_{\phi\phi}\Gamma^{\alpha}{}_{t\alpha} + \Gamma^{r}{}_{\phi\phi}\Gamma^{\alpha}{}_{r\alpha} + \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\alpha}{}_{\theta\alpha} \\ &= \Gamma^{t}{}_{\phi\phi}\Gamma^{i}{}_{ti} + \Gamma^{r}{}_{\phi\phi}\Gamma^{i}{}_{ri} + \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\theta\phi} \\ &= \left(3\dot{a}^2 + \kappa\right) r^2 \sin^2\theta - 1 - \sin^2\theta \ . \end{aligned} \tag{52}$$

The fourth term gives:

$$\begin{aligned}
{\Gamma^\rho}_{\phi\alpha}{\Gamma^\alpha}_{\rho\phi} &= {\Gamma^\rho}_{\phi t}{\Gamma^t}_{\rho\phi} + {\Gamma^\rho}_{\phi r}{\Gamma^r}_{\rho\phi} + {\Gamma^\rho}_{\phi\theta}{\Gamma^\theta}_{\rho\phi} + {\Gamma^\rho}_{\phi\phi}{\Gamma^\phi}_{\rho\phi} \\
&= {\Gamma^\phi}_{\phi t}{\Gamma^t}_{\phi\phi} + {\Gamma^\phi}_{\phi r}{\Gamma^r}_{\phi\phi} + {\Gamma^\phi}_{\phi\theta}{\Gamma^\theta}_{\phi\phi} + \left({\Gamma^t}_{\phi\phi}{\Gamma^\phi}_{t\phi} + {\Gamma^r}_{\phi\phi}{\Gamma^\phi}_{r\phi} + {\Gamma^\theta}_{\phi\phi}{\Gamma^\phi}_{\theta\phi}\right) \\
&= 2\left({\Gamma^\phi}_{\phi t}{\Gamma^t}_{\phi\phi} + {\Gamma^\phi}_{\phi r}{\Gamma^r}_{\phi\phi} + {\Gamma^\phi}_{\phi\theta}{\Gamma^\theta}_{\phi\phi}\right) \\
&= \left(2\dot{a}^2 + 2\kappa\right) r^2 \sin^2\theta - 2 \ . \qquad (53)
\end{aligned}$$

Thus we obtain:

$$R_{\phi\phi} = r^2 \sin^2\theta \left(2\kappa + 2\dot{a}^2 + a\,\ddot{a}\right) \ . \qquad (54)$$

Defining $\mathcal{A} \equiv 2\kappa + 2\dot{a}^2 + a\,\ddot{a}$ we may write the non-zero Ricci tensor components more succinctly as:

$$\begin{aligned}
R_{tt} &= -3\frac{\ddot{a}}{a} \ , \qquad (55) \\
R_{ii} &= g_{ii}\frac{\mathcal{A}}{a^2} \ . \qquad (56)
\end{aligned}$$

- For the third part of the question we are asked to calculate the Ricci scalar. This follows straightforwardly from equations (55)–(56):

$$\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} \\
&= g^{tt} R_{tt} + g^{ii} R_{ii} \\
&= 3\frac{\ddot{a}}{a} + g^{ii} g_{ii} \frac{\mathcal{A}}{a^2} \\
&= 3\frac{\ddot{a}}{a} + 3\frac{\mathcal{A}}{a^2} \\
&= \frac{6}{a^2}\left(\kappa + \dot{a}^2 + a\,\ddot{a}\right) \ . \qquad (57)
\end{aligned}$$

Exercise 2

Exploiting the results of the previous exercise, use the Einstein equations for the FLRW metric to derive the Friedmann equations. For simplicity set $\Lambda = 0$.

Solution 2

Since Exercise 3 requires $\Lambda > 0$ we will also assume this in the following solution. The Einstein field equations may be written as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \ , \qquad (58)$$

where

$$T_{\mu\nu} = (e + p) u_\mu u_\nu + p g_{\mu\nu} \ . \qquad (59)$$

We are in the comoving frame of the fluid, where $u^\alpha = (1, \underline{0})$ and $u_\alpha = (-1, \underline{0})$, and therefore:

$$\begin{aligned}
T_{tt} &= e \ , \qquad (60) \\
T_{ii} &= p g_{ii} \ . \qquad (61)
\end{aligned}$$

Considering the tt component of the Einstein field equations we obtain:

$$\begin{aligned} & R_{tt} - \frac{1}{2} g_{tt} R + \Lambda g_{tt} = 8\pi T_{tt} \\ \Longrightarrow \quad & -3\frac{\ddot{a}}{a} + \frac{3}{a^2}\left(\kappa + \dot{a}^2 + a\,\ddot{a}\right) - \Lambda = 8\pi e \\ \Longrightarrow \quad & \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left(8\pi e + \Lambda\right) - \frac{\kappa}{a^2} \ , \end{aligned} \tag{62}$$

which is the first Friedmann equation. We now consider the spatial component of the Einstein field equations:

$$\begin{aligned} & R_{ii} - \frac{1}{2} g_{ii} R + \Lambda g_{ii} = 8\pi T_{ii} \\ \Longrightarrow \quad & g_{ii}\frac{\mathcal{A}}{a^2} + g_{ii}\left(-\frac{1}{2}R + \Lambda - 8\pi p\right) = 0 \\ \Longrightarrow \quad & -2\frac{\ddot{a}}{a} - \frac{\kappa}{a^2} - \frac{\dot{a}^2}{a^2} + \Lambda - 8\pi p = 0 \qquad \text{(use equation (62))} \\ \Longrightarrow \quad & \frac{\ddot{a}}{a} = -\frac{4\pi}{3}\left(e + 3p\right) + \frac{\Lambda}{3} \ , \end{aligned} \tag{63}$$

which is the second Friedmann equation.

Exercise 3

Optional: Consider the case of an equation of state where $p = -e$ and $\Lambda > 0$. Derive the evolution equation for the scale factor. What type of universe is this?

Solution 3

In the comoving frame the stress-energy-momentum tensor of a perfect fluid may be written as:

$$\begin{aligned} T^{\mu}_{\ \nu} &= (e+p)u^{\mu}u_{\nu} + p\delta^{\mu}_{\nu} \\ &= \mathrm{diag}(-e, p, p, p) \ . \end{aligned} \tag{64}$$

The spatial component of the conservation equation ($\nabla_{\mu} T^{\mu}_{\ \nu} = 0$) trivially vanishes, implying uniform pressure. However, it is straightforward to show that the time component yields the fluid conservation equation:

$$\dot{e} + 3\frac{\dot{a}}{a}(e+p) = 0 \ . \tag{65}$$

For the given equation of state, this implies that $\dot{e} = 0$ and hence $e = e_0 =$ constant. Substituting this into the second Friedmann equation we obtain:

$$\begin{aligned} & \frac{\ddot{a}}{a} = \frac{8\pi}{3} e_0 + \frac{\Lambda}{3} \\ \Longrightarrow \quad & \ddot{a} = \mathcal{C} a \ , \end{aligned} \tag{66}$$

where $\mathcal{C} \equiv (\Lambda + 8\pi e_0)/3$. Now, since $\Lambda > 0$ and $e_0 \geq 0$, then this implies that $\mathcal{C} > 0$. Integrating equation (66) directly yields:

$$a(t) = c_1 \mathrm{e}^{\sqrt{C}t} + c_2 \mathrm{e}^{-\sqrt{C}t} \ , \tag{67}$$

where the integration constants c_1 and c_2 may be calculated from this equation and the first Friedmann equation as:

$$c_1 + c_2 = a_0 \ , \tag{68}$$

$$\mathcal{C} a_0^2 - \kappa = \dot{a}_0 \ , \tag{69}$$

where a_0 and $\dot{a}_0$ are the initial values of $a(t)$ and $\dot{a}(t)$ at $t = 0$. Equation (67) has a minimum at:

$$t_{\min} = \frac{1}{\sqrt{\mathcal{C}}} \ln \sqrt{\frac{c_2}{c_1}} \ , \tag{70}$$

and since $t \geq 0$ for the universe we know that if $c_2 > c_1$ then there exists a minimum value of $t > 0$. We assume c_1 and c_2 are both positive. Thus, if $c_1 > c_2$ the universe expands exponentially from $t = 0$. If, however, $c_2 > c_1$ then the universe contracts between $t = 0$ and $t_{\min}$, before expanding exponentially thereafter.