General Relativity: Solutions to exercises in Lecture XIII

Ziri Younsi

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Exercise 1

Consider the spherically symmetric static line element

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + B(r)\,\mathrm{d}r^2 + C(r)\,\mathrm{d}\Omega^2 \ , \tag{1}$$

and compute the expressions for the non-zero Christoffel symbols. Use this result to compute the 00 covariant component of the Einstein equations in vacuum, i.e. $R_{\mu\nu} = 0$.

Solution 1

- Whilst one may calculate the Christoffel symbol components directly, we will derive them from the Lagrangian for the metric. First let us write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\left(-A\,\dot{t}^2 + B\,\dot{r}^2 + C\,\dot{\theta}^2 + C\,\sin^2\theta\,\dot{\phi}^2\right) \ , \tag{2}$$

where the dependence of A, B and C on r has been omitted for brevity and an overdot denotes differentiation with respect to the affine parameter, λ. We now systematically derive the Euler-Lagrange equations of motion for each of the four components of our metric. For the t component:

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \ , \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{t}} = -A\,\dot{t} \ , \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = -A'\,\dot{r}\,\dot{t} - A\,\ddot{t} \ , \tag{5}$$

where primed quantities denote differentiation with respect to r. Thus from the Euler-Lagrange equations we obtain the geodesic equation of motion for t as

$$\ddot{t} = -\left(\frac{A'}{A}\right)\dot{r}\,\dot{t} \ . \tag{6}$$

This may be immediately compared to the geodesic equation of motion for t, yielding the non-zero Christoffel symbol components as

$$\Gamma^{t}{}_{tr} = \Gamma^{t}{}_{rt} = \frac{1}{2}\left(\frac{A'}{A}\right) \ . \tag{7}$$

Next we consider the r component of the Euler-Lagrange equations, yielding

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \frac{1}{2}\left(-A'\,\dot{t}^2 + B'\,\dot{r}^2 + C'\,\dot{\theta}^2 + C'\,\sin^2\theta\,\dot{\phi}^2\right) \ , && (8)\\
\frac{\partial \mathcal{L}}{\partial \dot{r}} &= B\,\dot{r}\,\dot{t} \ , && (9)\\
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) &= B'\,\dot{r}^2 + B\,\ddot{r} \ . && (10)
\end{aligned}$$

We may now write the geodesic equation of motion for r as

$$\ddot{r} = -\frac{1}{2}\left(\frac{A'}{B}\right)\dot{t}^2 - \frac{1}{2}\left(\frac{B'}{B}\right)\dot{r}^2 + \frac{1}{2}\left(\frac{C'}{B}\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{C'}{B}\right)\sin^2\theta\,\dot{\phi}^2 \ , \tag{11}$$

from which we directly obtain the Christoffel symbols as

$$\begin{aligned}
\Gamma^{r}{}_{tt} &= \frac{1}{2}\left(\frac{A'}{B}\right) \ , && (12)\\
\Gamma^{r}{}_{rr} &= \frac{1}{2}\left(\frac{B'}{B}\right) \ , && (13)\\
\Gamma^{r}{}_{\theta\theta} &= -\frac{1}{2}\left(\frac{C'}{B}\right) \ , && (14)\\
\Gamma^{r}{}_{\phi\phi} &= -\frac{1}{2}\left(\frac{C'}{B}\right)\sin^2\theta \ . && (15)
\end{aligned}$$

Now considering the θ component of the Euler-Lagrange equations we obtain

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} &= C\,\sin\theta\,\cos\theta\,\dot{\phi}^2 \ , && (16)\\
\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= C\,\dot{\theta}\,\dot{t} \ , && (17)\\
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) &= C'\,\dot{r}\,\dot{\theta} + C\,\ddot{\theta} \ . && (18)
\end{aligned}$$

We may now write the geodesic equation of motion for θ as

$$\ddot{\theta} = -\left(\frac{C'}{C}\right)\dot{r}\,\dot{\theta} + \sin\theta\,\cos\theta\,\dot{\phi}^2 \ , \tag{19}$$

from which we directly obtain the Christoffel symbols as

$$\begin{aligned}
\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} &= \frac{1}{2}\left(\frac{C'}{C}\right) \ , && (20)\\
\Gamma^{\theta}{}_{\phi\phi} &= -\sin\theta\cos\theta \ . && (21)
\end{aligned}$$

Finally, we consider the ϕ component of the Euler-Lagrange equations, obtaining

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} &= 0 \ , && (22) \\ \frac{\partial \mathcal{L}}{\partial \dot{\phi}} &= C \sin^2\theta \, \dot{t} \, \dot{\phi} \ , && (23) \\ \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) &= C' \sin^2\theta \, \dot{r} \, \dot{\phi} + C \sin 2\theta \, \dot{\theta} \, \dot{\phi} + C \sin^2\theta \, \ddot{\phi} \ . && (24) \end{aligned}$$

We may now write the geodesic equation of motion for ϕ as

$$\ddot{\phi} = -\left(\frac{C'}{C}\right) \dot{r} \, \dot{\theta} - 2 \cot\theta \, \dot{\theta} \, \dot{\phi} \ , \tag{25}$$

from which we directly obtain the remaining non-zero Christoffel symbols as

$$\begin{aligned} {\Gamma^{\phi}}_{r\phi} = {\Gamma^{\phi}}_{\phi r} &= \frac{1}{2}\left(\frac{C'}{C}\right) \ , && (26) \\ {\Gamma^{\phi}}_{\theta\phi} = {\Gamma^{\phi}}_{\phi\theta} &= \cot\theta \ . && (27) \end{aligned}$$

- For the second part of the question, recall the definition of the Riemann curvature tensor

$${R^{\alpha}}_{\beta\gamma\delta} = {\Gamma^{\alpha}}_{\beta\delta,\gamma} - {\Gamma^{\alpha}}_{\beta\gamma,\delta} + {\Gamma^{\mu}}_{\beta\delta}{\Gamma^{\alpha}}_{\mu\gamma} - {\Gamma^{\mu}}_{\beta\gamma}{\Gamma^{\alpha}}_{\mu\delta} \ . \tag{28}$$

The Ricci tensor is then defined as

$$\begin{aligned} R_{\beta\delta} &= {R^{\alpha}}_{\beta\alpha\delta} \\ &= {\Gamma^{\alpha}}_{\beta\delta,\alpha} - {\Gamma^{\alpha}}_{\beta\alpha,\delta} + {\Gamma^{\mu}}_{\beta\delta}{\Gamma^{\alpha}}_{\mu\alpha} - {\Gamma^{\mu}}_{\beta\alpha}{\Gamma^{\alpha}}_{\mu\delta} \ . && (29) \end{aligned}$$

The covariant 00 component may now be written as

$$\begin{aligned} R_{00} &= {\Gamma^{\alpha}}_{00,\alpha} - \cancelto{0}{{\Gamma^{\alpha}}_{0\alpha,0}} + {\Gamma^{\mu}}_{00}{\Gamma^{\alpha}}_{\mu\alpha} - {\Gamma^{\mu}}_{0\alpha}{\Gamma^{\alpha}}_{\mu 0} \\ &= {\Gamma^{\alpha}}_{00,\alpha} + {\Gamma^{\mu}}_{00}{\Gamma^{\alpha}}_{\mu\alpha} - {\Gamma^{\mu}}_{0\alpha}{\Gamma^{\alpha}}_{\mu 0} \\ &= {\Gamma^{r}}_{00,r} + {\Gamma^{r}}_{00}{\Gamma^{\alpha}}_{r\alpha} - {\Gamma^{\mu}}_{0\alpha}{\Gamma^{\alpha}}_{\mu 0} \\ &= {\Gamma^{r}}_{00,r} + {\Gamma^{r}}_{00}{\Gamma^{\alpha}}_{r\alpha} - {\Gamma^{\mu}}_{00}{\Gamma^{0}}_{\mu 0} - {\Gamma^{\mu}}_{0r}{\Gamma^{r}}_{\mu 0} \\ &= {\Gamma^{r}}_{00,r} + {\Gamma^{r}}_{00}{\Gamma^{\alpha}}_{r\alpha} - {\Gamma^{r}}_{00}{\Gamma^{0}}_{r0} - {\Gamma^{0}}_{0r}{\Gamma^{r}}_{00} \\ &= {\Gamma^{r}}_{00,r} + {\Gamma^{r}}_{00}{\Gamma^{\alpha}}_{r\alpha} - 2{\Gamma^{r}}_{00}{\Gamma^{0}}_{r0} \\ &= {\Gamma^{r}}_{00,r} + {\Gamma^{r}}_{00}\left({\Gamma^{0}}_{r0} + {\Gamma^{r}}_{rr} + {\Gamma^{\theta}}_{r\theta} + {\Gamma^{\phi}}_{r\phi}\right) - 2{\Gamma^{r}}_{00}{\Gamma^{0}}_{r0} \\ &= {\Gamma^{r}}_{00,r} + {\Gamma^{r}}_{00}\left({\Gamma^{r}}_{rr} + {\Gamma^{\theta}}_{r\theta} + {\Gamma^{\phi}}_{r\phi} - {\Gamma^{0}}_{r0}\right) \ . && (30) \end{aligned}$$

Substituting the values for the Christoffel symbol components into equation (30) we obtain, upon simplification

$$R_{00} = \frac{1}{2}\frac{A''}{B} + \frac{1}{4}\frac{A'}{B}\left[2\frac{C'}{C} - \frac{A'}{A} - \frac{B'}{B}\right] \ . \tag{31}$$

For completeness, the remaining non-zero covariant components of the Ricci tensor are

$$R_{11} = -\frac{1}{2}\frac{A''}{A} - \frac{C''}{C} + \frac{1}{4}\frac{A'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{2}\frac{C'}{C}\left(\frac{C'}{C} + \frac{B'}{B}\right) \ , \tag{32}$$

$$R_{22} = 1 - \frac{1}{2}\frac{C''}{B} + \frac{1}{4}\frac{C'}{B}\left(\frac{B'}{B} - \frac{A'}{A}\right) \ , \tag{33}$$

$$R_{33} = R_{22}\sin^2\theta \ . \tag{34}$$

Exercise 2

Using the Lagrangian

$$2\mathcal{L} = g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \ , \tag{35}$$

where an overdot corresponds to differentiation with respect to the proper time, show that the geodesic equations

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0 \ , \tag{36}$$

are equivalent to the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha}\right) = 0 \ . \tag{37}$$

Solution 2

Let us first calculate the first term in equation (37):

$$\frac{\partial\mathcal{L}}{\partial x^\gamma} = \frac{1}{2}g_{\alpha\beta,\gamma}\dot{x}^\alpha\dot{x}^\beta \ . \tag{38}$$

Now we consider the bracketed second term:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{x}^\gamma} &= \frac{1}{2}g_{\alpha\beta}\left(\delta^\alpha_\gamma\dot{x}^\beta + \dot{x}^\alpha\delta^\beta_\gamma\right)\\ &= \frac{1}{2}\left(g_{\gamma\beta}\dot{x}^\beta + \dot{x}^\alpha g_{\alpha\gamma}\right)\\ &= g_{\alpha\gamma}\dot{x}^\alpha \ .\end{aligned} \tag{39}$$

Finally, we differentiate equation (39) with respect to proper time, yielding:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\gamma}\right) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\alpha\gamma}\right)\dot{x}^\alpha + g_{\alpha\gamma}\ddot{x}^\alpha\\ &= \dot{x}^\delta g_{\alpha\gamma,\delta}\dot{x}^\alpha + g_{\alpha\gamma}\ddot{x}^\alpha\\ &= g_{\alpha\gamma,\delta}\dot{x}^\delta\dot{x}^\alpha + g_{\alpha\gamma}\ddot{x}^\alpha \ .\end{aligned} \tag{40}$$

Now we may write down the Euler-Lagrange equations, and solving for $\ddot{x}^\alpha$ we obtain

$$\begin{aligned}g_{\alpha\gamma}\ddot{x}^\alpha &= \frac{1}{2}g_{\alpha\beta,\gamma}\dot{x}^\alpha\dot{x}^\beta - g_{\alpha\gamma,\delta}\dot{x}^\delta\dot{x}^\alpha\\ &= \frac{1}{2}g_{\alpha\beta,\gamma}\dot{x}^\alpha\dot{x}^\beta - \frac{1}{2}\left(g_{\alpha\gamma,\delta}\dot{x}^\delta\dot{x}^\alpha + g_{\delta\gamma,\alpha}\dot{x}^\delta\dot{x}^\alpha\right) \ ,\end{aligned} \tag{41}$$

where we have made use of the symmetry under interchange of $\delta \leftrightarrow \alpha$ in the second term on the right hand side. Since δ is a dummy index we may relabel it as $\delta \to \beta$, yielding

$$g_{\alpha\gamma}\ddot{x}^{\alpha} = \frac{1}{2}\left(g_{\alpha\beta,\gamma} - g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}\right)\dot{x}^{\alpha}\dot{x}^{\beta} \ . \tag{42}$$

Multiplying both sides by $g^{\gamma\mu}$, using the identity $g_{\alpha\gamma}g^{\gamma\mu} = \delta^{\mu}_{\alpha}$ and bringing all terms to the left hand side we obtain

$$\ddot{x}^{\mu} + \frac{1}{2}g^{\gamma\mu}\left(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma}\right)\dot{x}^{\alpha}\dot{x}^{\beta} = 0 \ . \tag{43}$$

It is straightforward to confirm that the term multiplying $\dot{x}^{\alpha}\dot{x}^{\beta}$ is precisely ${\Gamma^{\mu}}_{\beta\alpha} = {\Gamma^{\mu}}_{\alpha\beta}$ and thus we obtain

$$\ddot{x}^{\mu} + {\Gamma^{\mu}}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = 0 \ , \tag{44}$$

which is the geodesic equation of motion, as required.

Exercise 3

Optional: Using the Einstein-Hilbert action

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\,R \ , \tag{45}$$

show that the application of a variational principle $\delta\mathcal{S} = 0$ yields the Einstein field equations in vacuum, i.e.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \ . \tag{46}$$

Solution 3

First we may write

$$\delta\mathcal{S} = 0 \iff \delta\int \mathrm{d}^4x\sqrt{-g}\,R = 0 \ . \tag{47}$$

Now let us vary $\sqrt{-g}$, yielding

$$\delta\left(\sqrt{-g}\right) = -\frac{\delta g}{2\sqrt{-g}} \ . \tag{48}$$

Now recall from Problem Sheet 7, Exercise 3, part 4, we proved the following result:

$$\left(\ln|g|\right)_{,\alpha} = g^{\mu\nu}g_{\mu\nu,\alpha} \ . \tag{49}$$

This implies that

$$g_{,\alpha} = g\,g^{\mu\nu}g_{\mu\nu,\alpha} \ , \tag{50}$$

and thus we may write δg as

$$\begin{aligned} \delta g &= g\,g^{\mu\nu}\,\delta g_{\mu\nu} \\ &= -g\,g_{\mu\nu}\,\delta g^{\mu\nu} \ . \end{aligned} \tag{51}$$

We may now write $\delta\left(\sqrt{-g}\right)$ as:

$$\begin{aligned}
\delta\left(\sqrt{-g}\right) &= \frac{g\, g_{\mu\nu}\, \delta g^{\mu\nu}}{2\sqrt{-g}} \\
&= -\frac{1}{2}\frac{(-g)}{\sqrt{-g}}\, g_{\mu\nu}\, \delta g^{\mu\nu} \\
&= -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu} \ . \qquad (52)
\end{aligned}$$

We must next consider the variation of the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$. We may write this as

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\, \delta R_{\mu\nu} \ . \qquad (53)$$

Substituting equations (52) and (53) into equation (47) yields:

$$\begin{aligned}
\delta \int \mathrm{d}^4x \sqrt{-g}\, R &= \int \mathrm{d}^4x \left[\delta\left(\sqrt{-g}\right) R + \sqrt{-g}\, \delta R\right] \\
&= \int \mathrm{d}^4x \sqrt{-g} \left[-\frac{1}{2} g_{\mu\nu}\, \delta g^{\mu\nu}\, R + \left(\delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}\right)\right] \\
&= \int \mathrm{d}^4x \sqrt{-g} \left[\delta g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, R\right) + g^{\mu\nu}\, \delta R_{\mu\nu}\right] \\
&= \int \mathrm{d}^4x \sqrt{-g} \left(\delta g^{\mu\nu}\, G_{\mu\nu} + g^{\mu\nu}\, \delta R_{\mu\nu}\right) = 0 \ , \qquad (54)
\end{aligned}$$

where $G_{\mu\nu} \equiv \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, R\right)$ is the Einstein tensor. It is now clear that in order for us to obtain the Einstein field equations in vacuum, the second term in brackets in equation (54) must vanish.

Let us now turn our attention to the variation of the Ricci tensor, $\delta R_{\mu\nu}$. First, recall the definition of the Riemann curvature tensor

$$R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}{}_{\nu\beta,\alpha} + \Gamma^{\mu}{}_{\rho\alpha} \Gamma^{\rho}{}_{\nu\beta} - \Gamma^{\mu}{}_{\nu\alpha,\beta} - \Gamma^{\mu}{}_{\rho\beta} \Gamma^{\rho}{}_{\nu\alpha} \ . \qquad (55)$$

Next, consider the variation of the Riemann curvature tensor:

$$\begin{aligned}
\delta R^{\mu}{}_{\nu\alpha\beta} = &\quad \partial_\alpha \left(\delta \Gamma^{\mu}{}_{\nu\beta}\right) + \left(\delta \Gamma^{\mu}{}_{\rho\alpha}\right) \Gamma^{\rho}{}_{\nu\beta} + \Gamma^{\mu}{}_{\rho\alpha} \left(\delta \Gamma^{\rho}{}_{\nu\beta}\right) \\
&- \partial_\beta \left(\delta \Gamma^{\mu}{}_{\nu\alpha}\right) - \left(\delta \Gamma^{\mu}{}_{\rho\beta}\right) \Gamma^{\rho}{}_{\nu\alpha} - \Gamma^{\mu}{}_{\rho\beta} \left(\delta \Gamma^{\rho}{}_{\nu\alpha}\right) \ . \qquad (56)
\end{aligned}$$

This expression can be written much more succinctly in terms of covariant derivatives. The first and fourth terms contains a partial derivative, so we consider the following:

$$\begin{aligned}
\nabla_\alpha \left(\delta \Gamma^{\mu}{}_{\nu\beta}\right) &= \partial_\alpha \left(\delta \Gamma^{\mu}{}_{\nu\beta}\right) + \Gamma^{\mu}{}_{\alpha\rho} \left(\delta \Gamma^{\rho}{}_{\nu\beta}\right) - \Gamma^{\rho}{}_{\alpha\nu} \left(\delta \Gamma^{\mu}{}_{\rho\beta}\right) - \Gamma^{\rho}_{\alpha\beta} \left(\delta \Gamma^{\mu}{}_{\nu\rho}\right) \ , \qquad (57) \\
\nabla_\beta \left(\delta \Gamma^{\mu}{}_{\nu\alpha}\right) &= \partial_\beta \left(\delta \Gamma^{\mu}{}_{\nu\alpha}\right) + \Gamma^{\mu}{}_{\beta\rho} \left(\delta \Gamma^{\rho}{}_{\nu\alpha}\right) - \Gamma^{\rho}{}_{\beta\nu} \left(\delta \Gamma^{\mu}{}_{\rho\alpha}\right) - \Gamma^{\rho}_{\alpha\beta} \left(\delta \Gamma^{\mu}{}_{\nu\rho}\right) \ . \qquad (58)
\end{aligned}$$

It immediately follows that the difference between equations (57) and (58) enables equation (56) to be written as

$$\delta R^{\mu}{}_{\nu\alpha\beta} = \nabla_\alpha \left(\delta \Gamma^{\mu}{}_{\nu\beta}\right) - \nabla_\beta \left(\delta \Gamma^{\mu}{}_{\nu\alpha}\right) \ . \qquad (59)$$

We may now calculate $\delta R_{\mu\nu}$ as follows:

$$\begin{aligned}
\delta R_{\nu\beta} &= \delta R^{\alpha}{}_{\nu\alpha\beta} \\
&= \nabla_\alpha \left(\delta \Gamma^{\alpha}{}_{\nu\beta}\right) - \nabla_\beta \left(\delta \Gamma^{\alpha}{}_{\nu\alpha}\right) \ , \qquad (60)
\end{aligned}$$

and thus we obtain upon relabelling indices ($\beta \leftrightarrow \nu$ followed by $\beta \rightarrow \mu$):

$$\delta R_{\mu\nu} = \nabla_\alpha \left(\delta\Gamma^\alpha{}_{\mu\nu}\right) - \nabla_\nu \left(\delta\Gamma^\alpha{}_{\mu\alpha}\right) \ . \tag{61}$$

We may now write the second term in brackets in equation (54) as:

$$\begin{aligned} g^{\mu\nu}\,\delta R_{\mu\nu} &= \nabla_\alpha \left(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu}\right) - \nabla_\nu \left(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\alpha}\right) \\ &= \nabla_\alpha \left(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu{}_{\mu\nu}\right) \ , \end{aligned} \tag{62}$$

where we have let $\alpha \leftrightarrow \nu$ in the second term. We may now write the second term in equation (54) as

$$\int \mathrm{d}^4x\sqrt{-g}\, g^{\mu\nu}\,\delta R_{\mu\nu} = \int \mathrm{d}^4x\sqrt{-g}\,\nabla_\alpha \left(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu{}_{\mu\nu}\right) \ . \tag{63}$$

To proceed further, recall Problem Sheet 7, Exercise 3, part 5, where we proved the following identity:

$$A^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,A^\alpha\right)_{,\alpha} \ . \tag{64}$$

We may define A^α from equation (63) as

$$A^\alpha = g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu{}_{\mu\nu} \ , \tag{65}$$

which enables us to rewrite equation (63) as

$$\begin{aligned} \int \mathrm{d}^4x\sqrt{-g}\, g^{\mu\nu}\,\delta R_{\mu\nu} &= \int \mathrm{d}^4x\,\partial_\alpha \left(\sqrt{-g}\,A^\alpha\right) \\ &= 0 \ , \end{aligned} \tag{66}$$

since this is a surface integral, yielding a constant boundary term, and by Stokes's Theorem vanishes. We may finally write

$$\delta\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\,\delta g^{\mu\nu}\, G_{\mu\nu} = 0 \ , \tag{67}$$

and so we may conclude that

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,R = 0 \ , \tag{68}$$

i.e. the Einstein field equations in vacuum, as required.