# General Relativity: Solutions to exercises in Lecture XII

Ziri Younsi

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## Exercise 1

Consider the stress-energy-momentum tensor of a perfect fluid

$$T^{\mu\nu} = (e+p)u^{\mu}u^{\nu} + p\,g^{\mu\nu} \ , \tag{1}$$

and its conservation equation

$$\nabla_{\mu}T^{\mu\nu} = 0 \ . \tag{2}$$

Show that the equations (2) lead to the Euler equations, i.e. to the equations of conservation of momentum

$$(e+p)\nabla_{\mathbf{u}}\mathbf{u} = -\left[\nabla p + (\nabla_{\mathbf{u}}\, p)\, \mathbf{u}\right] \ . \tag{3}$$

[Hint: use the projector $\mathbf{h} = \mathbf{g} + \mathbf{u}\,\mathbf{u}$]. Do equations (3) bear resemblance with the Newtonian Euler equations?

## Solution 1

First, let us write out the covariant derivative of the stress-energy-momentum tensor:

$$\nabla_{\mu}T^{\mu\nu} = (e+p)_{;\mu}u^{\mu}u^{\nu} + (e+p)\left(u^{\mu}{}_{;\mu}u^{\nu} + u^{\mu}u^{\nu}{}_{;\mu}\right) + p_{;\mu}g^{\mu\nu} \ . \tag{4}$$

Let us now us the projection tensor $h_{\alpha\nu} = g_{\alpha\nu} + u_{\alpha}u_{\nu}$ to project orthogonally to $\mathbf{u}$, yielding

$$\begin{aligned} h_{\alpha\nu}\nabla_{\mu}T^{\mu\nu} &= (e+p)_{;\mu}\cancel{[u^{\mu}u_{\alpha} + u_{\alpha}u^{\mu}(u_{\nu}u^{\nu})]} + (e+p)\Big[\cancel{\left(u^{\mu}{}_{;\mu}u_{\alpha} + u_{\alpha}u^{\mu}{}_{;\mu}(u_{\nu}u^{\nu})\right)} \\ &\quad + \left(u^{\mu}g_{\alpha\nu}u^{\nu}{}_{;\mu} + u_{\alpha}u^{\mu}u_{\nu}u^{\nu}{}_{;\mu}\right)\Big] + p_{,\mu}\delta^{\mu}_{\alpha} + p_{,\mu}u^{\mu}u_{\alpha} \\ &= (e+p)u^{\mu}u_{\alpha;\mu} + p_{,\alpha} + p_{,\mu}u^{\mu}u_{\alpha} \ , \end{aligned} \tag{5}$$

where we have used the fact that $u_{\nu}u^{\nu} = -1$ and $u_{\nu}u^{\nu}{}_{;\mu} = 0$. Since $h_{\alpha\nu}\nabla_{\mu}T^{\mu\nu} = 0$ we may now write

$$(e+p)u^{\mu}u_{\alpha;\mu} = -\left(p_{,\alpha} + p_{,\mu}u^{\mu}u_{\alpha}\right) \ , \tag{6}$$

which is equivalent to

$$(e+p)\nabla_{\mathbf{u}}\mathbf{u} = -\left[\nabla p + (\nabla_{\mathbf{u}}\, p)\, \mathbf{u}\right] \ , \tag{7}$$

as required.

In the Newtonian limit we may adopt the following approximations:

- $p \ll e$,
- $e \approx \rho_0$,
- $v^2 \ll 1$,
- $g_{00} = -(1+2\phi),\ \ |\phi| \ll 1$, where $\phi$ is the Newtonian potential.

We may immediately let $(e+p) \to \rho_0$, and through expanding the covariant derivative we obtain

$$\rho_0 \left[u^\mu u_{\alpha,\mu} - \Gamma^\beta{}_{\mu\alpha} u_\beta u^\mu\right] = -p_{,\alpha} - u_\alpha u^\mu p_{,\mu} \ . \tag{8}$$

We now take: (i) $u_\beta u^\mu \sim O(v^2)$ for $\beta \neq \mu$ and $u_\beta u^\mu = -1$ for $\beta = \mu$ and (ii) $u_\alpha u^\mu p_{,\mu} \sim O(v^2) \to 0$. With these in mind we obtain

$$\rho_0 \left[u^\mu u_{\alpha,\mu} + \Gamma^\beta{}_{\beta\alpha}\right] = -p_{,\alpha} \ . \tag{9}$$

Recall from Problem Sheet 7, Exercise 3, part 5 we derived the following expression:

$$\Gamma^\beta{}_{\beta\alpha} = \frac{1}{2} g^{\beta\delta} g_{\delta\beta,\alpha} \ . \tag{10}$$

Since spacetime is now flat and $g_{00} = -(1+2\phi)$, so $g^{00} = -(1/g_{00}) \approx -1$, since $|\phi| \ll 1$. This implies

$$\begin{aligned} \Gamma^\beta{}_{\beta\alpha} &= \Gamma^0{}_{0\alpha} \\ &\approx \frac{1}{2}(-1)\left[-(1+2\phi)\right]_{,\alpha} \\ &= \phi_{,\alpha} \ , \end{aligned} \tag{11}$$

hence we obtain

$$\rho_0 \left(u^\mu u_{\alpha,\mu} + \phi_{,\alpha}\right) = -p_{,\alpha} \ , \tag{12}$$

which may be rewritten as

$$u^\mu u_{\alpha,\mu} = -\frac{1}{\rho_0} p_{,\alpha} - \phi_{,\alpha} \ , \tag{13}$$

which is precisely the (Newtonian) incompressible Euler momentum equation with a constant and uniform density. This may be written more succinctly as follows. First define the specific thermodynamic work, $w$, where $w \equiv p/\rho_0$ and the gravitational acceleration $\mathbf{g} \equiv -\nabla\phi$. The material derivative is defined in general relativity as $\frac{\mathrm{D}}{\mathrm{D}\tau} = u^\mu \nabla_\mu$ and in the Newtonian regime as $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$. We may now rewrite equation (13) as

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla w + \mathbf{g} \ . \tag{14}$$

# Exercise 2

The stress-energy-momentum tensor of a scalar field $\Phi$ is defined as

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\partial_\mu\Phi\,\partial_\nu\Phi - \frac{1}{2} g_{\mu\nu}\,\partial_\alpha\Phi\,\partial^\alpha\Phi\right) \ . \tag{15}$$

Derive the expression for the conservation of energy and momentum (2) in this case. Interpret the results.

# Solution 2

We first write out the covariant derivative $\nabla^{\mu} T_{\mu\nu} = 0$ as follows

$$\nabla^{\mu}\left(4\pi T_{\mu\nu}\right) = \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \Phi_{,\mu}\Phi_{,\nu}{}^{;\mu} - \frac{1}{2} g_{\mu\nu}\left(\Phi_{,\alpha}\Phi^{,\alpha}\right)^{;\mu} \ . \tag{16}$$

Defining the differential portion of the third term as $\Delta = \left(\Phi_{,\alpha}\Phi^{,\alpha}\right)^{;\mu}$ we may write

$$\begin{aligned}
\Delta &= \Phi_{,\alpha}{}^{;\mu}\Phi^{,\alpha} + \Phi_{,\alpha}\Phi^{,\alpha;\mu} \\
&= g^{\alpha\beta} g_{\alpha\gamma}\Phi^{,\gamma;\mu}\Phi_{,\beta} + \Phi_{,\alpha}\Phi^{,\alpha;\mu} \\
&= \delta^{\beta}_{\gamma}\Phi^{,\gamma;\mu}\Phi_{,\beta} + \Phi_{,\alpha}\Phi^{,\alpha;\mu} \\
&= \Phi^{,\beta;\mu}\Phi_{,\beta} + \Phi_{,\alpha}\Phi^{,\alpha;\mu} \\
&= 2\Phi_{,\alpha}\Phi^{,\alpha;\mu} \ .
\end{aligned} \tag{17}$$

Equation (16) becomes

$$\begin{aligned}
\nabla^{\mu}\left(4\pi T_{\mu\nu}\right) &= \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \Phi_{,\mu}\Phi_{,\nu}{}^{;\mu} - \Phi_{,\alpha}\Phi^{,\alpha}{}_{;\nu} \\
&= \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \left(\Phi_{,\nu}{}^{;\mu} - \Phi^{,\mu}{}_{;\nu}\right)\Phi_{,\mu} \\
&= \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \left(g^{\alpha\mu}\Phi_{,\nu;\alpha} - g^{\beta\mu}\Phi_{,\beta;\nu}\right)\Phi_{,\mu} \\
&= \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \left(\Phi_{,\nu;\alpha} - \Phi_{,\alpha;\nu}\right) g^{\alpha\mu}\Phi_{,\mu} \\
&= \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \left(\Phi_{,\nu;\alpha} - \Phi_{,\alpha;\nu}\right)\Phi^{,\alpha} \ .
\end{aligned} \tag{18}$$

Since partial derivatives commute, and the covariant derivative of a scalar is simply the partial derivative, the second term in brackets vanishes and we may write

$$\nabla^{\mu}\left(4\pi T_{\mu\nu}\right) = \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} \ . \tag{19}$$

Since we assume $\Phi_{,\nu} \neq 0$ and $\Phi_{,\mu}{}^{;\mu} \equiv \Phi_{;\mu}{}^{;\mu}$ we may write the conservation of energy and momentum as

$$\Phi_{;\mu}{}^{;\mu} = 0 \ , \tag{20}$$

which is equivalent to

$$\Box\Phi = 0 \ . \tag{21}$$

Thus $\mathbf{\Phi}$ satisfies the wave equation for a scalar field in vacuum.

# Exercise 3

Show that the Einstein equations in vacuum reduce to

$$R_{\mu\nu} = 0 \ . \tag{22}$$

# Solution 3

Let us start from the definition of the Einstein Tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} \ . \tag{23}$$

In the presence of matter the Einstein field equations, $G_{\mu\nu} = 8\pi\, T_{\mu\nu}$ may be written as

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi T_{\mu\nu} \ . \tag{24}$$

Multiplying both sides of this equation by $g^{\mu\nu}$ yields

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R\, g^{\mu\nu} g_{\mu\nu} = 8\pi g^{\mu\nu} T_{\mu\nu} \ , \tag{25}$$

which simplifies to

$$R = -4\pi T \ , \tag{26}$$

where we have used the fact that $g^{\mu\nu} g_{\mu\nu} = 4$ and defined $T \equiv g^{\mu\nu} T_{\mu\nu}$. Substituting $R = -4\pi T$ back into equation (24) yields, upon simplification

$$R_{\mu\nu} = 8\pi \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \ . \tag{27}$$

In vacuum $T_{\mu\nu} = 0$, which implies $T = 0$, and thus we obtain

$$R_{\mu\nu} = 0 \ , \tag{28}$$

as required.