Parallel transport and the Riemann Curvature Tensor

Recall in lecture 3 the definition of parallel transport:

Consider a curve $\mathfrak{c}$ possessing a tangent vector T^a. A vector V^a at each point on $\mathfrak{c}$ is said to be parallely transported along $\mathfrak{c}$ if the conidition

$$T^a \nabla_a V^b = 0, \tag{1}$$

is satisfied along $\mathfrak{c}$. Expanding this from the definition of the covariant derivative (lecture 3)

$$T^a \partial_a V^b + T^a \Gamma^b{}_{ac} V^c = 0. \tag{2}$$

If we parametrise $\mathfrak{c}$ by the affine parameter λ such that $T^a = \frac{dX^a}{d\lambda}$ then it follows that

$$T^a \partial_a V^b = \frac{dX^a}{d\lambda}\frac{dV^b}{dX^a} = \frac{dV^b}{d\lambda}. \tag{3}$$

$$\therefore \quad \frac{dV^b}{d\lambda} + T^a \Gamma^b{}_{ac} V^c = 0. \tag{4}$$

Equation (4) implies the change in the parallely transported vector w.r.t. the affine parameter is zero if the connection vanishes ($\Gamma^b{}_{ac} = 0$). Furthermore, it follows that a vector defined at one point on $\mathfrak{c}$ uniquely defines the parallely transported vector everywhere else on $\mathfrak{c}$.

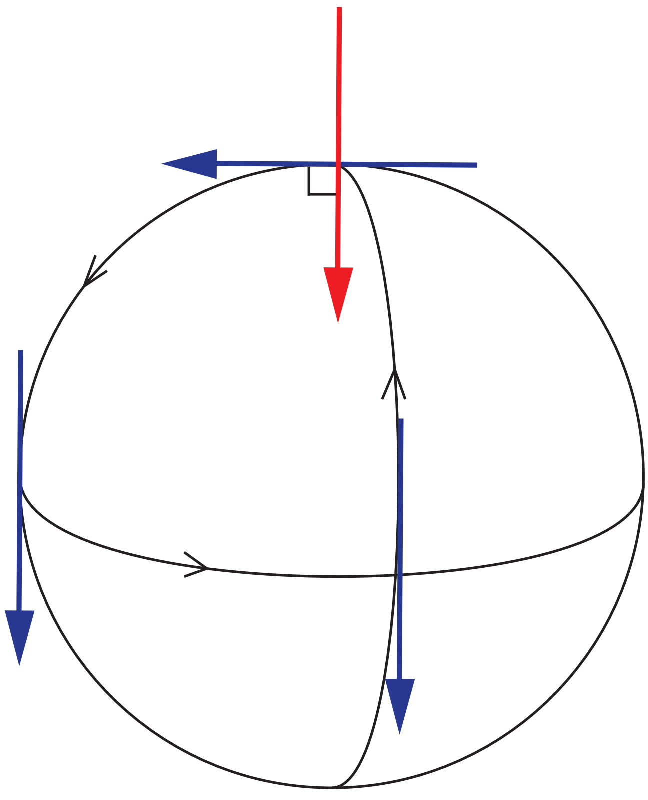


Figure 1: The vector (blue) is parallely transported anticlockwise around a closed loop defined by the spherical triangle. After one trip around the closed loop the new vector (red) is rotated by $\pi/2$ relative to the original vector. The deviation from the original vector is a measure of the curvature of the surface upon which the vector is parallely transported.

Parallel transport and its dependence on curvature

Consider a contravariant vector Y^b parallely transported (or displaced) along a curve ç. For infinitesimally small displacements it follows that

$$Y^b(X^a) \text{ at p} \tag{5}$$

$$Y^b(X^a + dX^a) \text{ at q} \tag{6}$$

Taylor expanding (6) gives

$$Y^b(X^a + dX^a) = Y^b(X^a) + \frac{\partial Y^b(X^a)}{\partial X^a} dX^a + \mathrm{O}(dX^2). \tag{7}$$

From the definition of parallel transport (1)

$$\begin{aligned} dX^a \nabla_a Y^b &= dX^a(\partial_a Y^b + \Gamma^b{}_{ac} Y^c) \\ \Rightarrow \frac{\partial Y^b}{\partial X^a} dX^a &= dX^a(\nabla_a Y^b - \Gamma^b{}_{ac} Y^c). \end{aligned} \tag{8}$$

It immediately follows that

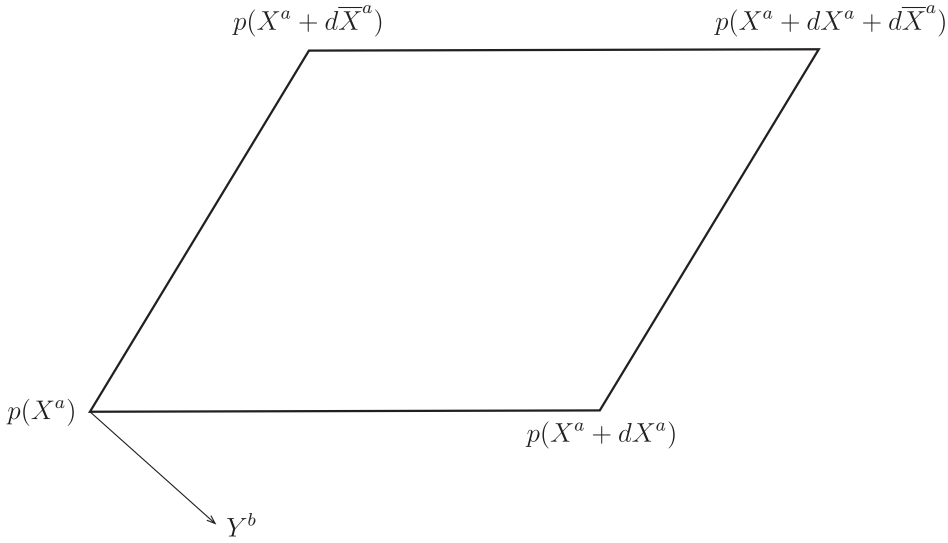
$$\begin{aligned} Y^b(X^a + dX^a) &= Y^b(X^a) + dX^a \nabla_a Y^b - dX^a \Gamma^b{}_{ac} Y^c + \mathrm{O}(dX^2) \\ &= Y^b(X^a) - dX^a \Gamma^b{}_{ac} Y^c + \mathrm{O}(dX^2), \end{aligned} \tag{9}$$

where the second term has vanished from the definition of parallel transport. It hence follows that the infinitesimal change in a parallely transported vector being moved from point p to point q is given

$$\begin{aligned} \delta Y^b &= Y^b(q) - Y^b(p) \\ &= -dX^a \Gamma^b{}_{ac} Y^c, \end{aligned} \tag{10}$$

to lowest order in dX.

Now let us consider the parallel transport of the vector Y^b around a closed loop, defined as a parallelogram for simplicity.



Let us transport Y^b from $p(X^a)$ to $p(X^a + dX^a)$. At $p(X^a + dX^a)$ the displaced vector is given by

$$Y^b + \delta Y^b = Y^b(p) - dX^a \Gamma^b{}_{ac} Y^c(p). \tag{11}$$

Next, transport the above transported vector to the point $p(X^a + dX^a + d\overline{X}^a)$, along $d\overline{X}^a$

$$\begin{aligned} \delta\overline{Y}^b &= -d\overline{X}^a \Gamma^b{}_{ac}(Y^c + \delta Y^c) \\ &= -d\overline{X}^a (\Gamma^b{}_{ac} + \Gamma^b{}_{ac,d} dX^d)(Y^c + \delta Y^c) + \mathrm{O}(dX^3). \end{aligned} \tag{12}$$

Using equation (10) to expand δY^c, i.e. using $\delta Y^c = -dX^d \Gamma^c{}_{ed} Y^e$

$$\begin{aligned} \delta\overline{Y}^b &= -d\overline{X}^a (\Gamma^b{}_{ac} + \Gamma^b{}_{ac,d} dX^d)(Y^c - dX^d \Gamma^c{}_{ed} Y^e) + \mathrm{O}(dX^3) \\ &= -d\overline{X}^a (\Gamma^b{}_{ac} Y^c - dX^d \Gamma^b{}_{ac} \Gamma^c{}_{ed} Y^e + dX^d \Gamma^b{}_{ac,d} Y^c) + \mathrm{O}(dX^3) \\ &= -d\overline{X}^a \Gamma^b{}_{ac} Y^c - d\overline{X}^a \Gamma^b{}_{ac,d} dX^d \; Y^c + d\overline{X}^a dX^d \Gamma^b{}_{ac} \Gamma^c{}_{ed} Y^e + \mathrm{O}(dX^3). \end{aligned} \tag{13}$$

Transporting the other way around the loop yields

$$\delta\overline{\overline{Y}}^b = -dX^a \Gamma^b{}_{ac} Y^c - dX^a \Gamma^b{}_{ac,d} d\overline{X}^d \; Y^c + dX^a d\overline{X}^d \Gamma^b{}_{ac} \Gamma^c{}_{ed} Y^e + \mathrm{O}(dX^3). \tag{14}$$

The change in the vector Y^b is given by

$$\begin{aligned} \Delta Y^b &= (Y^b + \delta\overline{Y}^b) - (Y^b + \delta\overline{\overline{Y}}^b) \\ &= (dX^a - d\overline{X}^a)\Gamma^b{}_{ac} Y^c - d\overline{X}^a \Gamma^b{}_{ac,d} dX^d \; Y^c + d\overline{X}^a dX^d \Gamma^b{}_{ac} \Gamma^c{}_{ed} Y^e \\ &+ dX^a \Gamma^b{}_{ac,d} d\overline{X}^d Y^c - dX^a d\overline{X}^d \Gamma^b{}_{ac} \Gamma^c{}_{ed} Y^e + \mathrm{O}(dX^3). \end{aligned} \tag{15}$$

Let us relabel the first term in (15) as $t^b = (dX^a - d\overline{X}^a)\Gamma^b{}_{ac} Y^c$. In the second term let $c \to e$ and interchange a and d. In the third term interchange a and d. For the fourth term let $c \to e$. The final term does not need to have any indices relabelled. The resulting relabelling of indices yields

$$\begin{aligned} \Delta Y^b &= t^b - d\overline{X}^d \Gamma^b{}_{de,a} dX^a \; Y^e + d\overline{X}^d dX^a \Gamma^b{}_{dc} \Gamma^c{}_{ea} Y^e \\ &+ dX^a \Gamma^b{}_{ae,d} d\overline{X}^d Y^e - dX^a d\overline{X}^d \Gamma^b{}_{ac} \Gamma^c{}_{ed} Y^e + \mathrm{O}(dX^3) \\ &= t^b + dX^a d\overline{X}^d Y^e \Big[\Gamma^b{}_{ae,d} - \Gamma^b{}_{de,a} + \Gamma^b{}_{cd} \Gamma^c{}_{ae} - \Gamma^b{}_{ac} \Gamma^c{}_{de} \Big] + \mathrm{O}(dX^3). \end{aligned} \tag{16}$$

The term in square brackets is precisely the Riemann curvature tensor $R_{ade}{}^b$. Consider the expression t^b. Transporting dX^a along $d\overline{X}^a$ and $d\overline{X}^a$ along dX^a one finds

$$\begin{aligned} dX^a \Gamma^b{}_{ac} d\overline{X}^c - d\overline{X}^a \Gamma^b{}_{ac} dX^c &= dX^a \Gamma^b{}_{ac} d\overline{X}^c - d\overline{X}^c \Gamma^b{}_{ca} dX^a \\ &= dX^a d\overline{X}^c (\Gamma^b{}_{ac} - \Gamma^b{}_{ca}) \\ &= 0. \end{aligned}$$

It thus follows from (16) the deviation of a parallely transported vector from the original vector, when transported around a closed loop is given by

$$\boxed{\Delta Y^b = R_{ade}{}^b dX^a d\overline{X}^d Y^e} \tag{17}$$

We can conclude that the Riemann curvature tensor determines the path dependence of parallel transport. Further, it follows that the failure of a parallely transported vector to coincide with the original (along a closed loop) is a measure of the curvature. If the surface is flat, the curvature vanishes and thus the change in a vector moved around a closed loop in Euclidean 3-space is zero. This is easily visualised by considering a euclidean triangle, as opposed to the spherical triangle in figure 1.