

General Relativity - Lecture 4

20/05/10

We looked at parallel transport, curvature, Riemann, and the Einstein field eqns.

Let us look again at geodesics. A geodesic is a curve whose tangent vector is parallelly transported along itself, so its tangent vector satisfies

$$\underline{T^a \nabla_a T^b = 0}$$

As before, ^{consider} a curve c , parametrised by λ , with tangent vector $T^a = \frac{dx^a}{d\lambda}$

$$\nabla_a T^b = \partial_a T^b + \Gamma^b_{ac} T^c$$

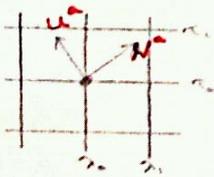
$$\begin{aligned} T^a \nabla_a T^b &= T^a \frac{\partial T^b}{\partial x^a} + \Gamma^b_{ac} T^a T^c \\ &= \frac{\partial x^a}{d\lambda} \frac{\partial T^b}{\partial x^a} + \Gamma^b_{ac} \frac{\partial x^a}{\partial \lambda} \frac{\partial x^c}{\partial \lambda} \\ &= \frac{\partial T^b}{\partial \lambda} + \Gamma^b_{ac} \frac{\partial x^a}{\partial \lambda} \frac{\partial x^c}{\partial \lambda} \\ &= \frac{\partial^2 x^b}{\partial \lambda^2} + \Gamma^b_{ac} \frac{\partial x^a}{\partial \lambda} \frac{\partial x^c}{\partial \lambda} = 0 \end{aligned}$$

hence:

$$\text{i.e. } \underline{\frac{\partial^2 x^a}{\partial \lambda^2} + \Gamma^a_{bc} \frac{\partial x^b}{\partial \lambda} \frac{\partial x^c}{\partial \lambda} = 0}, \text{ the Geodesic eqn. of motion.}$$

Geodesic deviation equation

Consider a surface spanned by geodesics $x^a = x^a(\tau, \lambda)$



The $\lambda = \text{const.}$ curves are parametrized by the affine parameter τ .

$$U^a = \frac{\partial x^a}{\partial \tau}, \text{ tangent vector to the geodesic.}$$

$$N^a = \frac{\partial x^a}{\partial \lambda}, \text{ displacement vector to (infinitesimally) nearby geodesic.}$$

We define the relative velocity and acceleration as

$$V^a = U^b \nabla_b N^a$$

$$a^a = U^c \nabla_c V^a$$

Using the definitions of U^a & N^a , commutativity of partial derivatives gives

$$\frac{\partial^2 x^b}{\partial \lambda \partial \tau} = \frac{\partial^2 x^b}{\partial \tau \partial \lambda}$$

$$\Rightarrow \frac{\partial x^a}{\partial \lambda} \frac{\partial}{\partial x^a} \left(\frac{\partial x^b}{\partial \tau} \right) = \frac{\partial x^a}{\partial \tau} \frac{\partial}{\partial x^a} \left(\frac{\partial x^b}{\partial \lambda} \right)$$

$$\Rightarrow N^a U^b_{;a} = U^a N^b_{;a}$$

$$\Rightarrow \underline{N^a U^b_{;a} = U^a N^b_{;a}} \quad (*) \text{ (since the connection is symmetric - exercise)}$$

$$a^a = U^c \nabla_c V^a$$

$$= U^c \nabla_c (U^b \nabla_b N^a)$$

$$= U^c \nabla_c (U^b N^a;_b)$$

$$= U^c \nabla_c (N^b U^a;_b) \quad \leftarrow \text{using (*)}$$

$$= U^c N^b;_c U^a;_b + U^c N^b U^a;_b;_c$$

From the definition of the Riemann curvature tensor

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) U^c = R_{bad}{}^c U^d$$

$$\Rightarrow U^c;_b;_a - U^c;_a;_b = R_{bad}{}^c U^d$$

$$\Rightarrow U^a;_b;_c - U^a;_c;_b = R_{bcd}{}^a U^d \quad (\text{relabel } c \leftrightarrow a)$$

$$\therefore a^a = U^c N^b;_c U^a;_b + U^c N^b (U^a;_c;_b + R_{bcd}{}^a U^d)$$

$$= U^c N^b;_c U^a;_b + U^c N^b U^a;_c;_b + \underbrace{U^c N^b R_{bcd}{}^a U^d}_{t^a}$$

$$= N^c U^b;_c U^a;_b + U^c N^b U^a;_c;_b + t^a \quad (\text{use (*) on the first term})$$

$$= N^c U^b;_c U^a;_b + U^b N^c U^a;_b;_c + t^a \quad (\text{relabel } b \leftrightarrow c \text{ in the 2nd term (dummy)})$$

$$= N^c \nabla_c (U^b U^a;_b) + t^a$$

$$= N^c \nabla_c (\underbrace{U^b \nabla_b N^a}_{\rightarrow 0, \text{ by parallel transport}}) + t^a$$

$$\Rightarrow \underline{a^a = N^b U^c U^d R_{bcd}{}^a}, \text{ the geodesic deviation equation}$$

- The acceleration vanishes if $R_{bcd}{}^a = 0$. Geodesics accelerate towards or away from each other depending on whether $R_{bcd}{}^a \gtrless 0$.
- Curvature gives rise to acceleration. But, conversely, acceleration gives rise to curvature.
- This has profound implications. In G.P. you could argue that driving a car, it accelerates, and the engine curves spacetime around the car, hence it moves. In Newtonian mechanics you have traction between the tyre and the road.

Quick Note

Recall $L = g_{ab} \dot{x}^a \dot{x}^b$. Due to affine parametrization we may set $L = \pm 1$ for particles with mass, and $L = 0$ for photons.

$$\text{if } \left. \begin{array}{l} (+, -, -, -) : L = +1 \\ (-, +, +, +) : L = -1 \end{array} \right\} \text{ particles with mass}$$

The Weak Field Limit

We will use the Lorentzian signature $(-, +, +, +)$.

Recall in Newtonian Physics

$$\frac{d^2 x^i}{dt^2} = - \frac{\partial \Phi}{\partial x^i} \quad (+) \quad (m\ddot{x} = -m\nabla\Phi)$$

Consider the metric

$$g_{ab} = \eta_{ab} + h_{ab},$$

i.e. the metric is a small perturbation of flat spacetime (Minkowski). To satisfy the weak field limit we require $|h_{ab}| \ll 1$.

For example, in our Solar System

$$|h_{ab}| \sim |h_{tt}| = \frac{M_\odot}{R_\odot} \sim 10^{-6} \quad (\text{deviation from flat spacetime measured on the surface of the sun})$$

$$g^{ab} = \eta^{ab} - h^{ab}$$

$\Gamma^a_{bc} = \delta^c_a$ to first order:

$$\begin{aligned} (\eta_{ab} + h_{ab})(\eta^{bc} - h^{bc}) &= (\eta_{ab}\eta^{bc} - \eta_{ab}h^{bc} + h_{ab}\eta^{bc} - h_{ab}h^{bc}) \\ &= \delta^c_a + h^c_a - h^c_a + O(h^2) = \delta^c_a + O(h^2) \\ &\approx \delta^c_a \end{aligned}$$

Let us work in coordinates $x^a = (ct, x, y, z)$, with small velocities $\frac{dx^i}{d\lambda} \ll \frac{dx^t}{d\lambda}$ ($i=x, y, z$).

$$\begin{aligned} \frac{d^2 x^a}{d\lambda^2} &= -\Gamma^a_{bc} \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} \\ &\approx -\Gamma^a_{tt} \left(\frac{dx^t}{d\lambda}\right)^2 \end{aligned}$$

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (g_{db,c} + g_{cd,b} - g_{bc,d})$$

$$\begin{aligned} \Gamma^a_{tt} &= \frac{1}{2} g^{ad} (g_{tb,t} + g_{td,t} - g_{tt,d}) \quad \leftarrow \text{staticity} \Rightarrow \text{partial derivatives vanish w.r.t. } t \\ &= -\frac{1}{2} g^{ad} g_{tt,d} \end{aligned}$$

$$\Gamma^t_{tt} = -\frac{1}{2} g^{td} g_{tt,d} = -\frac{1}{2} (\eta^{td} - h^{td}) (\eta_{tt,d} + h_{tt,d}) = 0$$

$$\begin{aligned} \Gamma^i_{tt} &= -\frac{1}{2} g^{id} g_{tt,d} = -\frac{1}{2} (\eta^{id} - h^{id}) (\eta_{tt,d} + h_{tt,d}) \quad (d \text{ must} = i) \\ &= -\frac{1}{2} (\eta^{ic} - h^{ic}) h_{tt,i} \quad (\eta_{tt,i} = 0) \\ &\approx -\frac{1}{2} \eta^{ic} h_{tt,i} \\ &= -\frac{1}{2} \frac{\partial h_{tt}}{\partial x^i} \end{aligned}$$

Given $\frac{dx^i}{d\lambda} \ll \frac{dx^t}{d\lambda}$, $\frac{d^2x^t}{d\lambda^2} = 0$ (Geodesic eqn. of motion) (1)

$$\Rightarrow \frac{dt}{d\lambda} = c, \text{ the speed of light.}$$

$$\frac{d^2x^i}{d\lambda^2} + \Gamma_{tt}^i \left(\frac{dx^t}{d\lambda}\right)^2 = 0$$

$$\Rightarrow \frac{d^2x^i}{d\lambda^2} = -\frac{c^2}{2} \frac{\partial h_{tt}}{\partial x^i}$$

Comparing this with the Newtonian equation (*) yields

$$\frac{c^2}{2} \frac{\partial h_{tt}}{\partial x^i} = -\frac{\partial \Phi}{\partial x^i}$$

$$\Rightarrow h_{tt} = -\frac{2\Phi}{c^2}$$

$$g_{tt} = h_{tt} + \eta_{tt}$$

$$= -\left(1 + \frac{2\Phi}{c^2}\right)$$

In Newtonian gravity, for a spherically symmetric mass distribution

$$\Phi(r) = -\frac{GM}{r}$$

$$\therefore \underline{g_{tt} = -\left(1 - \frac{2GM}{c^2 r}\right)}.$$

This is the g_{tt} -component of the most static, spherically symmetric metric, of signature $(-, +, +, +)$.

This formalism can be extended further, to compute the linearised Einstein tensor and thus the full linearised field equations. This gives rise to gravitational waves, which we will not discuss.

I may set a step-by-step exercise to derive this.

For the record, and as an exercise, the linearised Riemann tensor is

$$\underline{R_{abcd} = \frac{1}{2} (h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac})}.$$

The full linearised Einstein field equations are derived using this.

The Schwarzschild Solution

(5)

We seek solutions to the vacuum field equations $R_{ab} = 0$. These describe the external gravitational field of a static and spherically symmetric body, e.g. the Sun.

Recall Q9 of the problem sheet

The most general static, spherically symmetric metric with Lorentzian signature is

$$ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

We found the Christoffel symbols as

$$\Gamma_{tt}^t = \Gamma_{tr}^t = \frac{1}{2} \nu'$$

$$\Gamma_{rr}^r = \frac{1}{2} \lambda' \quad , \quad \Gamma_{\theta\theta}^r = -r e^{-\lambda}$$

$$\Gamma_{\phi\phi}^r = -\frac{1}{2} \nu' e^{-\lambda} \quad \Gamma_{\phi\phi}^\theta = -r e^{-\lambda} \sin^2\theta$$

$$\Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = \frac{1}{r} \quad \Gamma_{\phi\phi}^\theta = -\sin\theta \cos\theta$$

$$\Gamma_{r\phi}^\phi = \Gamma_{\phi r}^\phi = \frac{1}{r} \quad \Gamma_{\theta\phi}^\phi = \Gamma_{\phi\theta}^\phi = \cot\theta$$

The Ricci tensor is given by

$$R_{bd} = \Gamma_{bd,a}^a - \Gamma_{ad,b}^a + \Gamma_{ae}^a \Gamma_{bd}^e - \Gamma_{be}^a \Gamma_{ad}^e$$

$$R_{tt} = \Gamma_{tt,a}^a - \Gamma_{at,t}^a + \Gamma_{ae}^a \Gamma_{tt}^e - \Gamma_{te}^a \Gamma_{at}^e$$

$$\begin{aligned} \Gamma_{tt,a}^a &= \Gamma_{tt,r}^r = \frac{d}{dr} \left(\frac{1}{2} \nu' e^{2\nu-\lambda} \right) \\ &= \frac{1}{2} \nu'' e^{2\nu-\lambda} + \frac{1}{2} \nu' (\nu' - \lambda') e^{2\nu-\lambda} \end{aligned}$$

$$\begin{aligned} \Gamma_{ae}^a \Gamma_{tt}^e &= \Gamma_{ar}^r \Gamma_{tt}^r = (\Gamma_{tr}^t + \Gamma_{tr}^r + \Gamma_{\theta r}^\theta + \Gamma_{\phi r}^\phi) \Gamma_{tt}^r \\ &= \left(\frac{1}{2} (\lambda' + \nu') + \frac{2}{r} \right) \frac{1}{2} \nu' e^{2\nu-\lambda} \end{aligned}$$

$$\begin{aligned} \Gamma_{te}^a \Gamma_{at}^e &= \Gamma_{tt}^t \Gamma_{at}^t + \Gamma_{tr}^r \Gamma_{at}^r \quad (\text{no } \Gamma_{t\theta}^\theta = \Gamma_{t\phi}^\phi) \\ &= \Gamma_{tt}^r \Gamma_{tr}^t + \Gamma_{tr}^t \Gamma_{tt}^r = 2 \Gamma_{tt}^r \Gamma_{tr}^t \\ &= \frac{1}{2} \nu'^2 e^{2\nu-\lambda} \end{aligned}$$

$$R_{tt} = e^{\nu-\lambda} \left[\frac{1}{2} \nu'' + \frac{1}{4} \nu'^2 - \frac{1}{4} \nu' \lambda' + \frac{1}{r} \nu' \right]$$

Similarly,

$$R_{rr} = \left[-\frac{1}{2} \nu'' - \frac{1}{4} \nu'^2 + \frac{1}{4} \nu' \lambda' + \frac{1}{r} \lambda' \right]$$

$$R_{\theta\theta} = \left[1 - e^{-\lambda} + \frac{1}{2} r \lambda' e^{-\lambda} - \frac{1}{2} r \nu' e^{-\lambda} \right]$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$$

The vacuum field equations are $R_{ab}=0$:

$$\underline{R_{tt}=0}: \frac{1}{2} \nu'' + \frac{1}{4} \nu'^2 - \frac{1}{4} \nu' \lambda' + \frac{1}{r} \nu' = 0 \quad \text{①}$$

$$\underline{R_{rr}=0}: -\left(\frac{1}{2} \nu'' + \frac{1}{4} \nu'^2 - \frac{1}{4} \nu' \lambda' \right) + \frac{1}{r} \lambda' = 0 \quad \text{②}$$

①+②:

$$\frac{1}{r} (\lambda' + \nu') = 0 \Rightarrow \nu' = -\lambda'$$

$$\Rightarrow \nu = -\lambda + c$$

$$\Rightarrow e^\nu = e^c e^{-\lambda}$$

$$\Rightarrow \underline{e^\nu = e^{-\lambda}} \text{ if we rescale the time coordinate } t \rightarrow \sqrt{e^c} t$$

Substitute this into the $R_{\theta\theta}=0$ equation:

$$1 - e^\nu = \frac{1}{2} r \nu' e^\nu - \frac{1}{2} r \nu' e^\nu = 0$$

$$\Rightarrow 1 - e^\nu - r \nu' e^\nu = 0 \Rightarrow \frac{d}{dr} (r e^\nu) = 1$$

$$\Rightarrow r e^\nu = r - c \Rightarrow e^\nu = \left(1 - \frac{c}{r} \right) \quad \leftarrow c \text{ has units of mass}$$

From the weak field limit we know that $c = \frac{2GM}{c^2}$, the Schwarzschild radius.

$$\underline{\underline{\therefore ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \left(1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}}$$

the Schwarzschild metric.

This is the most important solution of the Einstein field equations.

Notes on the Schwarzschild metric

(1)

- Singular at $r=0$ and $r=2M$.
- $r=0$ is a true singularity, $r=2M$ is a coordinate singularity, it is the Schwarzschild radius

$$r_s = \frac{2GM}{c^2}$$

$$r_s \approx 3 \left(\frac{M}{M_\odot} \right) \text{ km}$$

- If a body's Schwarzschild radius exceeds its physical radius it will most likely collapse into a black hole.
- For normal astrophysical bodies, e.g. Sun, Earth the Schwarzschild radius is within the radius of the body, where the vacuum solution is no longer valid.
- The Schwarzschild solution describes a non-rotating black hole ($r < r_s$).
- It is also used to describe the exterior gravitational field around spherical mass distributions and is used to predict deviations from Newtonian gravity, e.g. the Sun.
G.P. Tests
 - deflection of light by the sun
 - gravitational redshift of light
 - perihelion precession of Mercury

The above tests work by solving or approximating the geodesics of the Schwarzschild metric.

Equatorial Geodesics

Assuming motion in the equatorial plane, we have $\theta = \pi/2 \Rightarrow \dot{\theta} = 0$

$$\mathcal{L} = -\left(1 - \frac{2m}{r}\right) \dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2m}{r}\right)} + r^2 \dot{\phi}^2$$

Since the signature is $(-, +, +, +)$, $\mathcal{L} = -1$ for particles (time-like)
 $\mathcal{L} = 0$ for photons (null)

Since \mathcal{L} is indep of t and ϕ , we will have two constants of motion from the E-L equations, E & L .

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{t}} \right) = 0 &\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{t}} = \text{const.} \Rightarrow -2 \left(1 - \frac{2m}{r}\right) \dot{t} = -2E \\ &\Rightarrow \underline{E = \left(1 - \frac{2m}{r}\right) \dot{t}} \quad (1) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = 0 &\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \text{const} \Rightarrow 2r^2 \dot{\phi} = \text{const} = 2L \\ &\Rightarrow \underline{L = r^2 \dot{\phi}} \quad (2) \end{aligned}$$

Substituting this ^{(1) & (2)} back into the Lagrangian:

$$\mathcal{L} = -\left(1 - \frac{2m}{r}\right)^{-1} E^2 + \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 + \frac{L^2}{r^2}$$

$$\begin{aligned} \left(1 - \frac{2m}{r}\right) \mathcal{L} &= -E^2 + \dot{r}^2 + \left(1 - \frac{2m}{r}\right) \frac{L^2}{r^2} \\ \Rightarrow \underline{\underline{\frac{1}{2} \dot{r}^2 + \frac{1}{2} \left(1 - \frac{2m}{r}\right) \left(\frac{L^2}{r^2} - 2\right) = \frac{1}{2} E^2}} \end{aligned}$$

"K.E." + "P.E." = 'Total Energy'

∴ Radial motion of a geodesic is analogous to a unit mass test particle with energy $E^2/2$.
The motion is determined by the effective potential:

$$\underline{\underline{V_{\text{eff}}(r) = \frac{1}{2} \left(1 - \frac{2m}{r}\right) \left(\frac{L^2}{r^2} - 2\right)}}$$

Newtonian, attractive,
denominator for r^2

$$V_{\text{eff}}(r) = \frac{L^2}{2r^2} - \frac{mL^2}{r^3} + \frac{mL}{r} - \frac{L}{2}$$

Centrifugal barrier
can't get too close in
an orbit

Normal Newtonian
potential term.
If particle, $-m$, attractive also.

With all of this we can easily compute the special orbit in the equatorial plane, parametrised as a function of r .

Consider the curve $\phi = \phi(r)$:

$$\begin{aligned}\frac{d\phi}{dr} &= \frac{d\phi}{d\lambda} \frac{d\lambda}{dr} \\ &= \frac{\dot{\phi}}{\dot{r}} = \frac{L}{r^2 \dot{r}}\end{aligned}$$

$$\text{Since } \dot{r}^2 = E^2 - \left(1 - \frac{2m}{r}\right) \left(\frac{L^2}{r^2} - \mathcal{L}\right)$$

$$\therefore \frac{d\phi}{dr} = \frac{L}{r^2} \left[E^2 - \left(1 - \frac{2m}{r}\right) \left(\frac{L^2}{r^2} - \mathcal{L}\right) \right]^{-1/2}$$

This is most easily done numerically, but by separation of variables one can integrate analytically, to evaluate in terms of elliptic functions.