

Four-volume

When working in General relativity we need to define an invariant volume element.

It is like a Jacobian and is derived from the properties of tensor densities, which we will not discuss here.

In G.R. the invariant volume element is given by

$$\sqrt{-g} d^4x,$$

where  $g = \det(g_{ab})$ .

- Cartesian

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, g = \det(g_{\mu\nu}) = -1$$

$$\Rightarrow \sqrt{-g} d^4x = dx dy dz$$

- Spherical Polar Coordinates

$$g_{ab} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}, g = -r^4 \sin^2\theta$$

$$\Rightarrow \sqrt{-g} d^4x = r^2 \sin\theta dr d\theta d\phi$$

- Kerr (Rotating) Black hole

$$g_{ab} = \begin{bmatrix} -\left(1-\frac{2r}{\Sigma}\right) & 0 & 0 & -\frac{2ar\sin^2\theta}{\Sigma} \\ 0 & \Sigma & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\frac{2ars\sin^2\theta}{\Sigma} & 0 & 0 & (r^2+a^2+\frac{2a^2-r^2\sin^2\theta}{\Sigma})\sin^2\theta \end{bmatrix}, g = -\Sigma^2 \sin^2\theta$$

$$\Sigma = (r^2+a^2\cos^2\theta)^2 \sin^2\theta$$

$$\Rightarrow \sqrt{-g} d^4x = \Sigma \sin\theta dr d\theta d\phi$$

(In the limit  $a \rightarrow 0$  (margin),  $\Sigma = (r^2 + a^2 \cos^2\theta) \rightarrow r^2$ , we recover spherical polar)

It follows that integrals of functions over spacetime are of the form

$$\int f(x^\mu) \sqrt{-g} d^4x$$

## Curvature

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### What is a manifold?

- It is a space, possibly curved, in which infinitesimally small regions appear flat.
- E.g., the Earth is approximately spherical, yet it appears flat to us.

### Strictly:

A manifold  $M$

- ①  $M$  is a set of points which can be mapped into  $\mathbb{R}^n$
- ② The mapping must be one-to-one.
- ③ If two mappings overlap, one of the two mappings must be a differentiable function of the other.

### Examples of manifolds

- $\mathbb{R}^n$  -  $n$ -dimensional flat space (real numbers)  $\xrightarrow{\quad}$   $\mathbb{R}^1$  - real line  
 $\mathbb{R}^2$  - real plane etc...
- $S^n$  -  $n$ -dimensional sphere  $\xrightarrow{\quad}$   $S^1$  - circle  
 $\xrightarrow{\quad}$   $S^2$  - sphere etc..

Different coordinate systems are suitable for different manifolds.

In flat space, e.g.  $\mathbb{R}^3$ , it makes sense to use  $\overset{(x,y,z)}{ds^2} = dx^2 + dy^2 + dz^2$ .

In  $S^2$ , on the surface of the unit sphere ( $r=1$ ), it makes sense to use  $(\theta, \phi)$  coordinates, and the metric  $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ .

Since manifolds are curved, this presents a problem. We must use non-Cartesian coordinates, since Cartesian coordinates only describe flat space. We must use the tensor methods developed so far.

The information about the curvature of the manifold is contained in the metric. The question is, how do we find it? It is not found in  $\Gamma^a_{bc}$ , as it is not a tensor and varies from point to point on the manifold.

It is contained in what we call the Riemann Curvature Tensor.

It contains first derivatives of the Christoffel symbols and hence second derivatives of the metric tensor. It is covariant. We will discuss this later.

## The Covariant Derivative

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In exercise 7 of problem sheet 1 it was asked to show that the partial derivative acting on a vector does not transform as a tensor.

This necessitates the construction of a covariant derivative, which enables us to form tensor equations in curved spacetimes.

The covariant derivative is denoted  $\nabla_a$ .

Just as the partial derivative <sup>action</sup> is abbreviated as  $\partial_a V^b \equiv V^b;_a$ , the covariant derivative is abbreviated as  $\nabla_a V^b \equiv V^b;_a$  (i.e. a semicolon).

$\nabla_a$  is a map on a manifold which takes a type  $(\tilde{n})$  tensor to a type  $(\tilde{n}+1)$  tensor.

### -Properties of the covariant derivative

#### ① Linearity

$$\nabla_c (\alpha A^{a_1 \dots a_m}_{b_1 \dots b_n} + \beta B^{a_1 \dots a_m}_{b_1 \dots b_n}) = \alpha \nabla_c A^{a_1 \dots a_m}_{b_1 \dots b_n} + \beta \nabla_c B^{a_1 \dots a_m}_{b_1 \dots b_n}, \alpha, \beta \in \mathbb{R}$$

#### ② Leibnitz rule

$$\nabla_c (A^{a_1 \dots a_m}_{b_1 \dots b_n} B^{a_1 \dots a_m}_{b_1 \dots b_n}) = (\nabla_c A^{a_1 \dots a_m}_{b_1 \dots b_n}) B^{a_1 \dots a_m}_{b_1 \dots b_n} + A^{a_1 \dots a_m}_{b_1 \dots b_n} \nabla_c B^{a_1 \dots a_m}_{b_1 \dots b_n}$$

#### ③ Commutation upon contraction

$$\nabla_c (A^{a_1 \dots, \gamma, \dots, a_m}_{b_1 \dots, \gamma, \dots, b_n}) = \nabla_c A^{a_1 \dots, \gamma, \dots, a_m}_{b_1 \dots, \gamma, \dots, b_n}$$

#### ④ No torsion

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f, \text{ for } f \in C^\infty \text{ or (set of infinitely differentiable functions).}$$

[We assume no torsion in G.R. - in other theories, e.g. Einstein-Cartan theory, this property is dropped]

#### ⑤ Action upon scalars

$$\nabla_a \gamma = \partial_a \gamma$$

These properties cannot be derived, they are demands we impose as part of the definition of the covariant derivative.

Recall,  $\partial_a \gamma^t$  transforms as a tensor,  $\partial_a V^b$  does not  
 $\nabla_a V^b - \partial_a V^b$  must be expressible in terms of  $V^b$  (by locality) (4)  
 $\nabla_a V^b - \partial_a V^b = C_{ac}^b V^c$

Condition ④  $\Rightarrow C_{ac}^b = C_{ca}^b$ .

Since  $\nabla_a V^b$  is a tensor by defn, and  $\partial_a V^b$  is not,  $C_{ac}^b$  cannot be a tensor.

$$\nabla_a V^b = \partial_a V^b + C_{ac}^b V^c \quad (*)$$

The RHS of (\*) must transform as a tensor, so the inhomogeneities (non-tensorial components) must cancel out - cf question 8, problem sheet 1.

Since  $\nabla_a \gamma^t = \partial_a \gamma^t$ , consider  $\gamma^t = V_b V^b$ :

$$\begin{aligned} \nabla_a (V_b V^b) &= (\nabla_a V_b) V^b + V_b (\nabla_a V^b) \\ &= (\nabla_a V_b) V^b + V_b (\partial_a V^b + C_{ac}^b V^c) \end{aligned}$$

$$\partial_a (V_b V^b) = (\partial_a V_b) V^b + V_b (\partial_a V^b)$$

$$\begin{aligned} \text{Equating: } & (\nabla_a V_b) V^b + C_{ac}^b V_b V^c = (\partial_a V_b) V^b \\ & \leftarrow \text{let } b=c \text{ (dummy indices)} \\ & (\nabla_a V_b) V^b = (\partial_a V_b) V^b - C_{ab}^c V_c V^b \\ & \Rightarrow \nabla_a V_b = \partial_a V_b - C_{ab}^c V_c. \end{aligned}$$

We have derived the action of the covariant derivative on contravariant and covariant vectors.

Metric Compatibility (Metricity condition)

$$\nabla_a g_{bc} = 0 \quad \text{and} \quad \nabla_b g_{ac} = 0$$

i.e.  $C_{bc}^a$  is uniquely fixed to be the Christoffel symbol  $\Gamma^a_{bc}$ , and  $\nabla_a$  is unique.

The proof of this will be a problem sheet exercise.

## Covariant derivative formulae

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$$\nabla_a V^b = \partial_a V^b + \Gamma_{ac}^b V^c$$

Rank 1, type (1) & (2)

Contravariant index gives +  $\Gamma$

Covariant index gives -  $\Gamma$

$$\nabla_a V_b = \partial_a V_b - \Gamma_{ab}^c V_c$$

$$\nabla_a T^{bc} = \partial_a T^{bc} + \Gamma_{ad}^b T^{dc} + \Gamma_{ad}^c T^{bd}$$

Rank 2, type (3), (2) and (1).

$$\nabla_a T_{bc} = \partial_a T_{bc} - \Gamma_{ab}^d T_{dc} - \Gamma_{ac}^d T_{bd}$$

$$\nabla_a T^b_c = \partial_a T^b_c + \Gamma_{ad}^b T^d_c - \Gamma_{ac}^d T^b_d$$

- General form (  $\nabla_a$  acting on a type (2) tensor )

$$\begin{aligned} \nabla_a T_{m_1 m_2 \dots m_p}^{m_1 m_2 \dots m_p} &= \partial_a T_{m_1 m_2 \dots m_p}^{m_1 m_2 \dots m_p} \\ &\quad + \Gamma_{ad}^{m_1} T_{n_1 n_2 \dots n_p}^{dm_2 \dots m_p} + \Gamma_{ad}^{m_2} T_{n_1 \dots n_{p-1} n_p}^{m_1 dm_3 \dots m_p} + \dots \\ &\quad - \Gamma_{an_1}^{\alpha} T_{\alpha n_2 \dots n_p}^{m_1 \dots m_p} - \Gamma_{an_2}^{\alpha} T_{\alpha n_1 \dots n_{p-1}}^{m_1 \dots m_p} - \dots \end{aligned}$$

## Parallel Transport

Consider a curve  $\mathcal{C}$  with tangent vector  $T^a$ . A vector  $V^a$  at each point on the curve  $\mathcal{C}$  is said to be parallelly transported along  $\mathcal{C}$  if

$$\underline{T^a \nabla_a V^b = 0},$$

is satisfied along  $\mathcal{C}$ .

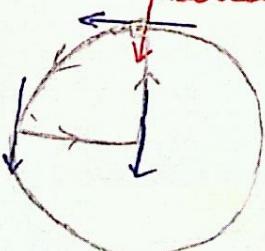
$$\text{i.e. } \underline{T^a \partial_a V^b + T^a \Gamma_{ac}^b V^c = 0}$$

Let us parametrize  $\mathcal{C}$  by  $\lambda$  s.t.  $T^a = \frac{dx^a}{d\lambda}$  then

$$T^a \partial_a V^b = \frac{dx^a}{d\lambda} \frac{dV^b}{dx^a} = \frac{dV^b}{d\lambda}$$

$$\therefore \underline{\frac{dV^b}{d\lambda} + T^a \Gamma_{ac}^b V^c = 0}$$

1st order ODE, unique solution for given initial value of  $V^a$ .  $\therefore$  a vector at one point on  $\mathcal{C}$  uniquely defines the parallelly transported vector everywhere else on  $\mathcal{C}$ .



→ Spherical triangle (a closed loop)

The vector parallelly transported around the loop is rotated  $90^\circ$  w.r.t. the original vector.

In fact, the failure of a vector to be parallelly transported onto itself is a measure of curvature.

A scalar  $\nabla_a W^a$  should remain unchanged when transported parallelly along  $C$ : (6)

$$T^a \nabla_a (\nabla_b W^b) = 0$$

$$\Rightarrow T^a \nabla_a (g_{bc} V^c W^b) = 0$$

$$\Rightarrow T^a (\underbrace{\nabla_a g_{bc}}_{=0 \text{ by metric by condition}}) + g_{bc} W^b (T^a \nabla_a V^c) + g_{bc} V^c (T^a \nabla_a W^b) = 0$$

$\uparrow$                                      $\uparrow$

$=0 \text{ by parallel transport defn.}$

### Riemann Curvature Tensor

We know  $\nabla_a, \nabla_b$  commute when acting on scalars, i.e.  $(\nabla_a \nabla_b - \nabla_b \nabla_a)V^c = 0$ . This is not true with vectors. Let's consider  $(\nabla_a \nabla_b - \nabla_b \nabla_a)V^c$ :

$$\begin{aligned} \nabla_a \nabla_b V^c &= \nabla_a (\partial_b V^c + \Gamma_{bd}^c V^d) \\ &= \nabla_a (\partial_b V^c) + \nabla_a (\Gamma_{bd}^c V^d) \\ &= \left[ \cancel{\frac{\partial_a (\partial_b V^c)}{1}} - \cancel{\frac{\Gamma_{ab}^d \partial_d V^c}{2}} + \cancel{\frac{\Gamma_{ad}^c \partial_b V^d}{4}} \right] \\ &\quad + \left[ \cancel{\partial_a (\Gamma_{bd}^c V^d)} - \cancel{\frac{\Gamma_{ab}^e \Gamma_{ed}^c V^d}{3}} + \Gamma_{ad}^c \Gamma_{be}^d V^e \right] \end{aligned}$$

Similarly ( $b \leftrightarrow a$ ):

$$\begin{aligned} \nabla_b \nabla_a V^c &= \left[ \cancel{\frac{\partial_b (\partial_a V^c)}{1}} - \cancel{\frac{\Gamma_{ab}^d \partial_d V^c}{2}} + \cancel{\frac{\Gamma_{bd}^c \partial_a V^d}{5}} \right] \\ &\quad + \left[ \cancel{\partial_b (\Gamma_{ad}^c V^d)} - \cancel{\frac{\Gamma_{ab}^e \Gamma_{ed}^c V^d}{3}} + \Gamma_{bd}^c \Gamma_{ae}^d V^e \right] \end{aligned}$$

$$\cancel{\partial_a (\Gamma_{bd}^c V^d)} = \Gamma_{bd,a}^c V^d + \cancel{\Gamma_{bd}^c \partial_a V^d}$$

$$\cancel{\partial_b (\Gamma_{ad}^c V^d)} = \Gamma_{ad,b}^c V^d + \cancel{\Gamma_{ad}^c \partial_b V^d}$$

Let:  $d \leftrightarrow e$

$$\begin{aligned} (\nabla_a \nabla_b - \nabla_b \nabla_a)V^c &= \Gamma_{bd,a}^c V^d - \Gamma_{ad,b}^c V^d + \Gamma_{ad}^c \Gamma_{be}^d V^e - \Gamma_{bd}^c \Gamma_{ae}^d V^e \\ &= \Gamma_{bd,a}^c V^d - \Gamma_{ad,b}^c V^d + \Gamma_{ae}^c \Gamma_{bd}^e V^d - \Gamma_{be}^c \Gamma_{ad}^e V^d \\ &= (\Gamma_{bd,a}^c - \Gamma_{ad,b}^c + \Gamma_{ae}^c \Gamma_{bd}^e - \Gamma_{be}^c \Gamma_{ad}^e) V^d \\ &= \underline{\underline{R_{bad}^c V^d}} \end{aligned}$$

$$\underline{\underline{R_{bad}^c}} = (\Gamma_{bd,a}^c - \Gamma_{be}^c \Gamma_{ad}^e) - (a \leftrightarrow b)$$

swap  $a \leftrightarrow b$  in the previous bracketed term

$R_{bad}^c$  is the Riemann curvature tensor

$$[(\nabla_a \nabla_b - \nabla_b \nabla_a) V_c = R_{abc}^d V_d]$$

The contraction of the Riemann tensor is the Ricci tensor,  $R_{ab}$ :

$$\underline{R_{ab} = R_{acb}^c}$$

The trace of the Ricci tensor yields the Ricci scalar

$$\underline{R = R^a_a = g^{bc} R_{bc}}$$

The Riemann tensor has 4 indices and  $4^4 = 256$  components, but it has many symmetries. It contains only 20 independent components.  
 $\approx \frac{n^2}{12}(n^2-1)$  in  $n$ -dimension.

### Properties

Consider  $R_{abcd}$ , the lowered Riemann tensor contracted with the metric, i.e.  $R_{abcd} = g_{ae} R^e{}_{bcd}$ :

$$R_{abcd} = -R_{abdc} = -R_{bacd}$$

$$R_{abcd} = R_{cdab}$$

$$R_{a[bcd]} = 0$$

$$R_{ab} = R_{ba} \quad (\text{Ricci tensor is symmetric})$$

$$\underline{\nabla_{[a} R_{bc]de} = 0}, \text{ the Bianchi identity.}$$

[The proof of all of these will be set as a problem in the problem sheet]

I will demonstrate the geometrical significance of the Riemann curvature tensor in the lectures.

The fact that the failure of a vector parallelly transported around a closed loop to coincide with the original vector coincides with curvature of the manifold, as well as this dependence being directly proportional to Riemann curvature tensor will be given as a handout later. It is too lengthy to derive in the time we have.

## Einstein Tensor and the Einstein Field Equations

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The Bianchi identity states that

$$\nabla_{[a} R_{bc]de} = 0$$

$$\Rightarrow \nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0$$

$$\Rightarrow \nabla_a R_{bcde} + \nabla_b R_{cade} - \nabla_c R_{bade} = 0$$

Multiply by  $g^{ae}$  (right multiply)

$$\nabla_a R_{bcd}^a + \nabla_b R_{cad}^a - \nabla_c R_{bad}^a = 0 \quad \leftarrow \text{Recall } R_{cad}^a \sim R_{cd} \text{, Ricci tensor}$$

$$\Rightarrow \nabla_a R_{bcd}^a + \nabla_b R_{cd} - \nabla_c R_{bd} = 0$$

Multiply by  $g^{bd}$ :

$$\nabla_a R_c^a + \nabla_b R_c^b - \nabla_c R = 0 \quad \leftarrow a, b \text{ are dummy indices}$$

$$\Rightarrow 2 \nabla_b R_c^b - \nabla_c R = 0$$

$$\Rightarrow \nabla_b R_c^b - \frac{1}{2} \nabla_c R = 0 \quad \leftarrow \text{Twice contracted Bianchi identity}$$

Multiply by  $g^{ac}$ :

$$\nabla_b R^{ab} - \frac{1}{2} \nabla_c g^{ac} R = 0 \quad \leftarrow c \text{ is a dummy index}$$

$$\Rightarrow \nabla_b (R^{ab} - \frac{1}{2} g^{ab} R) = 0$$

OR

$$\nabla^b (R_{ab} - \frac{1}{2} g_{ab} R) = 0$$

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \text{ Einstein tensor (symmetric)}$$

$$\nabla_a G^{ab} = 0$$

$G_{ab}$  contains components of the spacetime, inc. its curvature.

Recall lecture 2, where the stress-energy-momentum tensor  $T^{ab}$  satisfied

$$\nabla_a T^{ab} = 0.$$

The Einstein field equations are

$$\underline{\underline{G_{ab} = 8\pi T^{ab}}}$$

The LHS describes spacetime (i.e. geometry)

The RHS describes matter.

These equations are non-linear!

- matter tells spacetime how to curve

- spacetime tells matter how to move

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$$G_{ab} = 8\pi T_{ab}$$

$$\Rightarrow R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab}$$

Multiply by  $g^{ab}$ :

$$R = \frac{1}{2}(4)R = 8\pi T \Rightarrow R = -8\pi T$$

$$\therefore R_{ab} = 8\pi(T_{ab} - \frac{1}{2}g_{ab}T)$$

interior of a star, dense E.O.S., pressure etc.

In the presence of a cosmological constant  $\Lambda$ :

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \text{ yields (after calculation)}$$

$$R_{ab} = 8\pi(T_{ab} - \frac{1}{2}g_{ab}T + \Lambda g_{ab})$$

$T_{ab}$  describes matter. With no cosmological constant and in vacuum:

$$R_{ab} = 0, \text{ the vacuum field equations.}$$

### Geodesics & the Deviation Equation

- Do this next week!