

- 1905 - Einstein introduced special relativity, a theory of space and time. Assumes speed of light in vacuum is constant, for all observers, regardless of state of motion. The consequence of this upper bound leads to effects like length-contraction, time dilation etc..
- 10 years later, in 1915, Einstein proposed General Relativity, a geometric theory of gravitation. A unification of special relativity and Newtonian gravitation, it describes gravity as a geometric property of space and time, which is referred to in G.R. as spacetime (space and time are treated as part of the same 4D manifold)

Notation

G.R. has its root in differential geometry, being a geometrical theory of spacetime. Position and placement of indices is crucial.

$$(t, x, y, z) \rightarrow (x^0, x^1, x^2, x^3) \quad \begin{matrix} 0 \rightarrow t \\ (1, 2, 3) \rightarrow (x, y, z) \end{matrix}$$

OR $(t, r, \theta, \phi) \rightarrow (u^t, u^r, u^\theta, u^\phi)$

Throughout G.R., summation convention is used.

$$a = (a^1, a^2, \dots, a^n),$$

$$b = (b^1, b^2, \dots, b^n)$$

$$a^\alpha b_\alpha = \sum_{\alpha=1}^n a^\alpha b_\alpha,$$

Drop summation symbol wherever possible. Sum over twice repeated indices.
Never repeat an index more than twice

$$(t, r, \theta, \phi) \xrightarrow{\text{transformation}} (t', r', \theta', \phi')$$

Paired variables refer to a new coordinate system.

The partial derivative is written

$$\frac{\partial f}{\partial x^\alpha} = \partial_\alpha f \equiv f, \alpha$$

Vectors & Tensors

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Vectors

- Contravariant vector ('normal' vector)

Indicated by superscript, is a column vector.

Transforms as

$$V^a = \frac{\partial x^a}{\partial x'^b} V^b$$

- Covariant vector (used essential in G.R.)

Indicated by subscript, is a row vector

Transforms as

$$W_b = \frac{\partial x^c}{\partial x'^b} W_c$$

Consider $\mu = V^a W_a$

$$\mu = V^a W_a = - \frac{\partial x^a}{\partial x'^b} \frac{\partial x^c}{\partial x'^a} V^b W_c = S^c_b V^b W_c = V^c W_c \stackrel{\text{sum over dummy indices}}{\equiv} V^a W_a = \mu$$

$$\Rightarrow V^a W_a = \mu \quad (\mu = \mu \Rightarrow \mu \text{ is scalar})$$

Contravariant and covariant vectors have inverse transformation laws, they combine to give a number. They are said to be dual

Example - S.R.

Consider momentum equation covariant 4-momentum $(\bar{\beta} = \bar{v}/c)$

$$p_a = m u_a$$

$$\gamma = \frac{1}{\sqrt{1-\bar{\beta}^2}}$$

$$4\text{-velocity } u^a = \begin{pmatrix} \gamma c \\ \gamma \bar{v} \end{pmatrix} \equiv \gamma c \left(\begin{pmatrix} 1 \\ \bar{\beta} \end{pmatrix} \right), u_a = \gamma c (1 - \bar{\beta})$$

In flat spacetime, define $g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$u_a = g_{ab} u^b = \gamma c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \left(\begin{pmatrix} 1 \\ \bar{\beta} \end{pmatrix} \right) = \gamma c (1 - \bar{\beta}), \text{ in covariant form}$$

$$p_a = \gamma m c \left(\begin{pmatrix} 1 \\ \bar{\beta} \end{pmatrix} \right), p_a = \gamma m c (1 - \bar{\beta})$$

$$\text{Since } E = \gamma m c^2, \bar{p} = \gamma m \bar{v} = \gamma m c \bar{\beta}$$

$$p_a = \left(\begin{pmatrix} E/c \\ \bar{p} \end{pmatrix} \right), p_a = \left(\begin{pmatrix} E \\ \bar{p} \end{pmatrix} \right), -\bar{p}$$

$$p_a p^a = \gamma^2 m^2 c^2 (1 - \bar{\beta}) \left(\begin{pmatrix} 1 \\ \bar{\beta} \end{pmatrix} \right) = \gamma^2 (1 - \bar{\beta}^2) m^2 c^2 = m^2 c^2$$

$$p_a p^a = \left(\begin{pmatrix} E \\ \bar{p} \end{pmatrix} \right) \left(\begin{pmatrix} E/c \\ \bar{p} \end{pmatrix} \right) = \frac{E^2}{c^2} - |\bar{p}|^2 \Rightarrow m^2 c^2 = \frac{E^2}{c^2} - |\bar{p}|^2 \Rightarrow E^2 = |\bar{p}|^2 c^2 + m^2 c^2$$

If $E^2 = m^2 c^2$, $E = m c$

$$\text{If } v \ll c: E = p^0 = \gamma m c^2 = m c^2 (1 - \frac{v^2}{c^2})^{-1/2} = m c^2 + \frac{1}{2} m v^2 + O(v^4)$$

$E + \text{K.E.}$ (total energy content)

Example - G.R.

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We define angular velocity in G.R. as

$$\omega = \frac{u_\phi}{u_t} = \frac{\dot{\phi}}{c} = -\frac{g_{\phi\phi} u^t}{g_{tt} u^t}$$

However, angular momentum is defined as

$$l = -\frac{u_\phi}{u_t}, \quad \Delta \equiv -\frac{g_{t\phi} u^t + g_{\phi\phi} u^t}{g_{tt} u^t + g_{\phi\phi} u^t} = -\frac{g_{tt} + \sqrt{g_{tt} g_{\phi\phi}}}{g_{tt} + \sqrt{g_{tt} g_{\phi\phi}}}$$

which is covariant and must be expressed in terms of the metric components -
we will come to this later (of the spacetime)

For the Kerr metric.

Tensors

Tensor of type (m^n) - m superscripts, n subscripts

Rank of tensor is $m+n$

Transformation laws of type (m^n) tensor

$$T'^{a_1 a_2 \dots a_m}_{b_1 b_2 \dots b_n} = \frac{\partial x'^{a_1}}{\partial x^c_1} \dots \frac{\partial x'^{a_m}}{\partial x^c_m} \frac{\partial x'^{d_1}}{\partial x^b_1} \dots \frac{\partial x'^{d_n}}{\partial x^b_n} T^{c_1 c_2 \dots c_m}_{d_1 d_2 \dots d_n}$$

A contravariant tensor is of type (1^0) and rank 1.

A covariant tensor is of type (0^1) and rank 1.

Rank

0 scalar 1, n, e, ...

1 vector (...) or (:) e.g. T^a , T_a e.g. (u^a, p_a)

2 matrix $\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ e.g. T^{ab} , T^a_b , T_{ab} (T^{ab})

3 3D matrix $\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ e.g. T^{abc} , T^a_{bc} , T_{abc} ... (Γ^a_{bc})

4 matrix within a matrix $\begin{pmatrix} (\cdot) & (\cdot) \\ (\cdot) & (\cdot) \end{pmatrix}$ e.g. T^{abcd} , ... (R^{abcd})

5 matrix within 3D matrix \dots Tensor composition, e.g. $\Gamma^a_{bc} g^{de}$
 \downarrow deal with this next!

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Tensor Algebra

Addition - tensors of same type and index structure can be added

$$V^a_{bc} + W^a_{bc} = T^a_{bc} \quad (\text{not } V^a + W_a !)$$

Composition - combine type (\overline{n}) & type (\underline{p}) to get a type $(\frac{m+p}{n+p})$ tensor

Let $V^a = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $W_b = \begin{pmatrix} 3 & 4 \end{pmatrix}$

$$T^a_b = V^a W_b = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$

Contraction - Take type (\overline{n}) tensor. Sum over upper and lower indices, resulting in a type $(\frac{m-1}{n-1})$ tensor

$$T^{a_1 a_2 \dots s \dots a_m}_{b_1 b_2 \dots s \dots b_n} = U^{a_1 \dots a_m}_{b_1 \dots b_n} \quad \begin{matrix} (a_1 \dots a_m) \text{ is } m-1 \text{ indices} \\ (b_1 \dots b_n) \text{ is } n-1 \text{ indices} \end{matrix}$$

e.g. $T^a_a = T^1_1 + T^2_2 = 3 + 8 = 11$

Trace - consider a type $(!)$, rank 2 tensor S^a_b . Trace defined as,

$$\text{tr } S = S^a_a$$

Make sure LHS and RHS of tensor equations agree!

e.g. $K = R_{abcd} R^{abcd}$ is valid, $K = R_{abcd} F^a_a$ is not!

Decomposition

Decompose any rank-2, type $(\overline{2})$ tensor as T^{ab}

$$T^{(ab)} = \frac{1}{2} (T^{ab} + T^{ba}) \quad \text{symmetrisation}$$

$$T^{[ab]} = \frac{1}{2} (T^{ab} - T^{ba}) \quad \text{antisymmetrisation}$$

T^{ab} symmetric if $T^{ab} = T^{(ab)}$ $\Rightarrow T^{ab} = T^{ba}$

T^{ab} antisymmetric if $T^{ab} = T^{[ab]}$ $\Rightarrow T^{ab} = -T^{ba}$

$$\therefore T^{ab} = T^{(ab)} + T^{[ab]}$$

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For general type (r) , type r tensor

$$T^{(a_1 a_2 \dots a_r)} = \frac{1}{r!} \sum (\text{all permutations } a_i \text{ to } a_r)$$

$$T^{[a_1 a_2 \dots a_r]} = \frac{1}{r!} \sum (\text{alternating sum over all permutations } a_i \text{ to } a_r)$$

Example - T^{abc}

$$T^{(abc)} = \frac{1}{3!} (T^{abc} + T^{acb} + T^{bca} + T^{bac} + T^{cab} + T^{cba})$$

$$T^{[abc]} = \frac{1}{3!} (T^{abc} - T^{acb} + T^{bca} - T^{bac} + T^{cab} - T^{cba})$$

Levi-Civita - Alternating Tensor

$$\epsilon^{a_1 a_2 \dots a_n} = \begin{cases} 0 & \text{if } 2 \text{ or } 2 \text{ of } a_1 \dots a_n \text{ are equal} \\ +1 & \text{if even permutation of } a_1 \dots a_n \\ -1 & \text{if odd permutation of } a_1 \dots a_n \end{cases}$$

$$\text{in 2D } \epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We often work in 3D, so we are interested in ϵ^{ijk} .

Example

$$-\bar{a} \times \bar{b} \times \bar{c} = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c}$$

$$\begin{aligned} [\bar{a} \times (\bar{b} \times \bar{c})]^i &= \epsilon^{ijk} a_j \epsilon^{klm} b_l c_m = \epsilon^{jik} \epsilon^{klm} a_j b_l c_m \\ &= (\delta^{il} \delta^{jm} - \delta^{im} \delta^{jl}) a_j b_l c_m \\ &= a_j b^i c^j - a_j b^j c^i = (a_j c^j) b^i - (a_j b^j) c^i \\ &= [(\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c}]^i \end{aligned}$$

$$-\nabla \times (f \bar{a}) = \nabla f \times \bar{a} + f \nabla \times \bar{a}$$

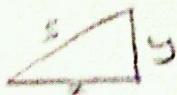
$$\begin{aligned} [\nabla \times (f \bar{a})]^i &= \epsilon^{ijk} \partial_j (f a_k) = \epsilon^{ijk} \frac{\partial f}{\partial x_j} a_k + \epsilon^{ijk} f \frac{\partial a_k}{\partial x_j} \\ &= \epsilon^{ijk} \frac{\partial f}{\partial x_j} a_k + f \epsilon^{ijk} \frac{\partial a_k}{\partial x_j} \\ &= [\nabla f \times \bar{a} + f \nabla \times \bar{a}]^i \end{aligned}$$

The Metric & Geodesics

How to measure distance in curved spacetime?

In Cartesian

$$s^2 = x^2 + y^2$$



The gradient is $\partial_x \partial_y = 0$

$ds^2 = dx^2 + dy^2$, the line element for 2D flat space

In G.R., distance is defined upon introducing the metric tensor g_{ab} by the line element

$$\boxed{ds^2 = g_{ab} dx^a dx^b}$$

The metric tensor can raise or lower indices

$$\begin{cases} V^a = g^{ab} V_b \\ V_a = g_{ab} V^b \end{cases} \rightarrow \text{maps between contravariant and covariant vectors}$$

Assume g_{ab} symmetric, otherwise $g_{ab}dx^a dx^b \neq g_{ab}dx^b dx^a$

-Cartesian 3D

$$ds^2 = dx^2 + dy^2 + dz^2, \text{ signature } (+, +, +) \text{ or } 3$$

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (x^1, x^2, x^3) = (x, y, z)$$

-Minkowski - metric for flat spacetime

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2, \text{ signature } (+, -, -, -) \text{ or } -2$$

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ especially reserved symbol for Minkowski.}$$

For a photon, the interval connecting two points in spacetime, $ds^2 = 0$.

-Spherical Polar coordinates

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$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$dx = \frac{dr}{r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi$$

$$= \sin \theta \cos \phi dr + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi$$

$$dy = \sin \theta \sin \phi dr + r \cos \theta \sin \phi d\theta + r \sin \theta \cos \phi d\phi$$

$$dz = \cos \theta dr - r \sin \theta d\theta$$

$$ds^2 = \sin^2 \theta \cos^2 \phi dr^2 + r^2 \sin^2 \theta \cos^2 \phi d\theta^2 + r^2 \sin^2 \theta \sin^2 \phi d\phi^2$$

$$+ 2 r \sin \theta \cos \theta \cos^2 \phi d\theta - 2 r^2 \sin \theta \cos \theta \cos \phi d\phi - 2 r \sin \theta \sin \phi \cos \phi d\phi$$

$$dy^2 = \sin^2 \theta \sin^2 \phi dr^2 + r^2 \cos^2 \theta \sin^2 \phi d\theta^2 + r^2 \sin^2 \theta \cos^2 \phi d\phi^2$$

$$+ 2 r \sin \theta \cos \theta \sin^2 \phi dr d\theta + 2 r^2 \sin \theta \cos \theta \sin^2 \phi d\theta d\phi$$

$$+ 2 r^2 \sin^2 \theta \cos^2 \phi d\phi$$

$$dz^2 = \cos^2 \theta dr^2 + r^2 \sin^2 \theta d\theta^2 - 2 r \sin \theta \cos \theta d\theta$$

$$dx^2 + dy^2 = \sin^2 \theta dr^2 + r^2 \cos^2 \theta d\theta^2 + r^2 \sin^2 \theta d\phi^2 + 2 r \sin \theta \cos \theta dr d\theta$$

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

$$\therefore ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

This is identical to the Cartesian, but we cannot show this until we compute the geodesic eqns of motion for test particles
 i.e. straight lines in flat space are straight lines, what are they in spherical polar coordinates? It turns out they are the same ..

$$\text{Remember } \tau = \frac{u^b}{u^c}, \quad l = -\frac{u^b}{u^c}. \quad ds^2 = -dt^2 + du^2 + r^2 d\theta^2$$

$$\text{For spherical polar coordinates } l = -\frac{g_{ab} u^a}{g_{cc} u^c} = -\tau \frac{g_{ab}}{g_{cc}}$$

$$= -\tau \frac{r^2 \sin^2 \theta}{1} = -\tau r^2 \sin^2 \theta$$

$$\Rightarrow l = \tau r^2 \sin^2 \theta \quad (\text{comes across third later})$$

Curve

Defined as $\mathcal{C}(\lambda) = \begin{pmatrix} x^1(\lambda) \\ \vdots \\ x^n(\lambda) \end{pmatrix} = x^\alpha(\lambda) \Rightarrow \mathcal{C}(\lambda) = x^\alpha$

Parametric curve by λ . λ -affine parameter

Tangent Vector

Tangent vector to the curve \mathcal{C} is given as

$$T^\alpha = \frac{\partial x^\alpha}{\partial \lambda} \quad (\equiv \dot{x}^\alpha)$$

The length of the curve is

$$\int ds = \int \frac{ds}{d\lambda} d\lambda = \int \sqrt{g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}} d\lambda \quad (\text{recall } ds^2 = g_{ab} dx^a dx^b)$$

$$\Rightarrow \int ds = \int \sqrt{g_{ab} \dot{x}^a \dot{x}^b} d\lambda$$

Q - For 2 fixed points in spacetime, which curves are the shortest? How do we extremise?

Geodesics

'straightest' possible lines. Strictly, a curve whose path extremises

$$\int ds = \int \sqrt{g_{ab} \dot{x}^a \dot{x}^b} d\lambda$$

Curve parametrised s.t. $g_{ab} \dot{x}^a \dot{x}^b = g_{ab} T^a T^b = 1$ (affine)

Extremisation of functional naturally leads to the Lagrangian

$$L = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b$$

Euler-Lagrange equations:

$$\boxed{\frac{\partial L}{\partial x^c} - \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^c} \right) = 0}$$

Geodesic equations of motion

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Taking $L = g_{ab} \dot{x}^a \dot{x}^b$ we have,

$$\frac{\partial L}{\partial x^c} = \frac{\partial g_{ab}}{\partial x^c} \dot{x}^a \dot{x}^b = g_{ab,c} \dot{x}^a \dot{x}^b$$

$$\frac{\partial L}{\partial \dot{x}^c} = g_{ab} \left(\frac{\partial x^a}{\partial x^c} \dot{x}^b + \dot{x}^a \frac{\partial x^b}{\partial x^c} \right)$$

$$= g_{ab} (\delta^a_c \dot{x}^b + \dot{x}^a \delta^b_c)$$

$$= g_{bc} \dot{x}^b + g_{ac} \dot{x}^a = 2g_{ac} \dot{x}^a \quad (\text{a \& b are dummy indices})$$

$$\frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^c} \right) = 2 \frac{dg_{ac}}{d\lambda} \dot{x}^a + 2g_{ac} \ddot{x}^a$$

$$= 2 \frac{dg_{ac}}{d\lambda} \frac{dx^b}{dx^c} \frac{dx^a}{d\lambda} + \dots$$

$$= 2 \frac{dg_{ac}}{dx^b} \frac{dx^b}{d\lambda} \frac{dx^a}{d\lambda} + \dots$$

$$= 2g_{ac,b} \dot{x}^a \dot{x}^b + 2g_{ac} \ddot{x}^a$$

$$= g_{ac,b} \dot{x}^a \dot{x}^b + g_{bc,a} \dot{x}^a \dot{x}^b + 2g_{ac} \ddot{x}^a$$

$$\therefore (g_{ac,b} + g_{bc,a} - g_{ab,c}) \dot{x}^a \dot{x}^b + 2g_{ac} \ddot{x}^a = 0$$

Multiply both sides by g^{ab}

$$\underbrace{g^{ad} g_{ac} \dot{x}^a}_{\delta^d_a} + \frac{1}{2} g^{ad} (g_{ac,b} + g_{bc,a} - g_{ab,c}) \dot{x}^a \dot{x}^b = 0$$

$$\Rightarrow \ddot{x}^d + \frac{1}{2} g^{ad} (g_{ac,b} + g_{bc,a} - g_{ab,c}) \dot{x}^a \dot{x}^b = 0$$

Relabel $a \rightarrow a, b \rightarrow c, c \rightarrow b, d \rightarrow 0$

$$\ddot{x}^a + \frac{1}{2} g^{ad} (g_{db,c} + g_{dc,b} - g_{bc,d}) \dot{x}^b \dot{x}^c = 0$$

$\Gamma^a_{bc} \leftarrow \text{Christoffel symbols (2nd kind, w.r.t. t)}$

$$\therefore \ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0 \quad \leftarrow \text{Geodesic}$$

$$\frac{d^2x^a}{dt^2} = -\Gamma_{bc}^a \dot{x}^b \dot{x}^c \quad (10)$$

Newtonian

$$\frac{d^2x^i}{dt^2} = -\frac{\partial \Phi}{\partial x^i} \quad \text{from } m\ddot{x}^i = -m\nabla\Phi(\bar{x})$$

Since the Christoffel symbols contain derivatives of the metric, the Geodesic equation of motion is the general relativistic analogue of the Newtonian form i.e., instead of gravitational potential we use the metric components of the spacetime itself, the derivatives of which are contained in the Christoffel symbols.

So we see, in G.R., that the curvature of the spacetime (the spacetime is) given to the gravitational potential. We will discuss this much more later on.

END of Lecture 1.

Geodesic Eqs of motion - Shortcut to the Christoffel symbols

Euclidean Space in spherical polar

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

$$L = g_{ab} \dot{x}^a \dot{x}^b = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2$$

$$\frac{\partial L}{\partial r} = 2\dot{r}^2 + 2r\sin^2\theta \dot{\phi}^2$$

$$\frac{\partial L}{\partial \theta} = 2\dot{r} \cdot \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = 2\dot{r}$$

$$\Rightarrow \boxed{\dot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 = 0} \quad (1)$$

Compare with $\ddot{x}^a + \Gamma_{bc}^a \dot{x}^b \dot{x}^c = 0$

$$\begin{aligned} \Gamma_{rr}^r &= -r \\ \Gamma_{\theta\theta}^r &= -r\sin^2\theta \end{aligned}$$

$$\frac{\partial L}{\partial \theta} = 2r^2 \sin\theta \cos\theta \dot{\phi}^2$$

$$\frac{\partial L}{\partial \phi} = 2r^2 \dot{\theta} \cdot \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 4r\dot{r}\dot{\theta} + 2r^2 \dot{\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{2}{r} \dot{r}\dot{\theta} - \sin\theta \cos\theta \dot{\phi}^2 = 0} \quad (2) \quad \begin{aligned} \Gamma_{r\theta}^{\theta} &= \Gamma_{\theta r}^{\theta} = \frac{1}{r} \\ \Gamma_{\phi\theta}^{\theta} &= -\sin\theta \cos\theta \end{aligned}$$

$$\frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{const}$$

$$\therefore 2r^2 \sin\theta \dot{\phi} = C \Rightarrow \boxed{r^2 \sin\theta \dot{\phi} = \text{const}}$$

$$\boxed{\dot{\phi} + \frac{2}{r} \dot{r}\dot{\phi} + 2\omega\theta\dot{\phi} = 0} \quad \Gamma_{r\phi}^{\phi} = \Gamma_{\phi r}^{\phi} = \frac{1}{r}, \quad \Gamma_{\theta\phi}^{\phi} = 2\omega\theta = \Gamma_{\phi\theta}^{\phi}$$

$\therefore l = r^2 \sin\theta$
 $r = \frac{l}{\sqrt{1 + \frac{4\omega^2}{l^2} \sin^2\theta}}$

inf
spac
time

From $r^2 \sin\theta \phi = \text{const}$

$$\Rightarrow r = r_0, \theta = \theta_0, \phi = \text{const}$$

from ① $\Rightarrow \dot{\phi}^2 = 0 \Rightarrow \phi = \text{const}$, similarly for ②

(r, θ, ϕ) are fixed : we have straight lines!

The coordinate system in spherical polar coordinates and Cartesian are the same, since the geodesics, or shortest paths, are identical.

i.e. \equiv Euclidean 3-space

