Hydrodynamics & Magnetohydrodynamics: Solutions to Exercises in Lecture I

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Exercise 1

Prove the Newtonian H-theorem, that is,

$$\frac{\partial f_0}{\partial t} = \Gamma\left(f_0\right) = 0\,, \tag{1.1}$$

where f_0 is the equilibrium distribution function. In particular, show that the condition (1.1) is fully equivalent to the condition

$$f_0\left(\vec{\mathbf{u}}_2'\right) f_0\left(\vec{\mathbf{u}}_1'\right) - f_0\left(\vec{\mathbf{u}}_2\right) f_0\left(\vec{\mathbf{u}}_1\right) = 0\,, \tag{1.2}$$

where the functional dependence of the distribution function on position $(\vec{\mathbf{x}})$ and time (t) has been neglected for brevity, i.e., $f_0\left(\vec{\mathbf{u}}\right) := f_0\left(t, \vec{\mathbf{x}}, \vec{\mathbf{u}}\right)$. By introducing the following shorthand notation for the distribution functions before and after collision: $f_{1,2} := f_0\left(t, \vec{\mathbf{x}}, \vec{\mathbf{u}}_{1,2}\right)$ and $f_{1,2}' := f_0\left(t, \vec{\mathbf{x}}, \vec{\mathbf{u}}_{1,2}'\right)$, equation (1.2) may be written more succinctly as

$$f_2'\, f_1' - f_2\, f_1 = 0\,, \tag{1.3}$$

where subscripts "1" and "2" refer to the particles undergoing the collision and unprimed/primed variables refer to quantities before/after collision.

Solution 1

First we introduce the definition of the Boltzmann H-function,

$$H(t) = \int d^3u\, f\left(t, \vec{\mathbf{u}}\right) \ln\left[f\left(t, \vec{\mathbf{u}}\right)\right]\,, \tag{1.4}$$

where $f\left(t, \vec{\mathbf{u}}\right)$ is the distribution function evaluated at time t, which satisfies

$$\frac{\partial f\left(t, \vec{\mathbf{u}}_1\right)}{\partial t} = \int d^3u_2 \int d\Omega\, \sigma(\Omega)\left|\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2\right|\left(f_2'\, f_1' - f_2\, f_1\right)\,. \tag{1.5}$$

Differentiation of the H-function with respect to time yields:

$$\frac{dH(t)}{dt} = \int d^3u\, \frac{\partial f\left(t, \vec{\mathbf{u}}\right)}{\partial t}\left\{1 + \ln\left[f\left(t, \vec{\mathbf{u}}\right)\right]\right\}\,. \tag{1.6}$$

It follows from the derivative of the H-function that if $\partial f/\partial t = 0$ then $dH/dt = 0$ also, i.e., $dH/dt = 0$ is a necessary and sufficient condition for $\partial f/\partial t = 0$.

To prove equivalence with the aforementioned condition in the question, we must prove equivalence with $dH/dt = 0$. First, substitute equation (1.5) into equation (1.6), yielding:

$$\frac{dH(t)}{dt} = \int d^3u_1 \int d^3u_2 \int d\Omega\, \sigma(\Omega)\, |\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2|\, (f_2'\, f_1' - f_2\, f_1)\, (1 + \ln f_1)\,. \tag{1.7}$$

The cross-section, $\sigma(\Omega)$, is invariant under interchange of particles undergoing collision (i.e. relabelling $1 \leftrightarrow 2$), thus we may also write

$$\frac{dH(t)}{dt} = \int d^3u_2 \int d^3u_1 \int d\Omega\, \sigma(\Omega)\, |\vec{\mathbf{u}}_2 - \vec{\mathbf{u}}_1|\, (f_1'\, f_2' - f_1\, f_2)\, (1 + \ln f_2)\,. \tag{1.8}$$

Since for these last two equations we may assume without loss of generality that $\int d^3u_1 \int d^3u_2 = \int d^3u_2 \int d^3u_1$ and $|\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2| = |\vec{\mathbf{u}}_2 - \vec{\mathbf{u}}_1|$, adding equations (1.7) and (1.8) together yields:

$$2\frac{dH(t)}{dt} = \int d^3u_1 \int d^3u_2 \int d\Omega\, \sigma(\Omega)\, |\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2|\, (f_2'\, f_1' - f_2\, f_1)\, [2 + \ln (f_1 f_2)]\ . \tag{1.9}$$

We next note that for each collision there is a corresponding inverse collision with the same cross-section, i.e., the cross-section is invariant under interchange of unprimed and primed quantities. Exploiting this fact, we may now rewrite equation (1.9) as:

$$2\frac{dH(t)}{dt} = \int d^3u_1' \int d^3u_2' \int d\Omega'\, \sigma'(\Omega')\, |\vec{\mathbf{u}}_1' - \vec{\mathbf{u}}_2'|\, (f_2\, f_1 - f_2'\, f_1')\, [2 + \ln (f_1' f_2')]\ . \tag{1.10}$$

Upon noting that $d^3u_1\, d^3u_2 = d^3u_1'\, d^3u_2'$, $\sigma(\Omega) = \sigma'(\Omega')$ and $|\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2| = |\vec{\mathbf{u}}_1' - \vec{\mathbf{u}}_2'|$, equation (1.10) may be rewritten as:

$$2\frac{dH(t)}{dt} = -\int d^3u_1 \int d^3u_2 \int d\Omega\, \sigma(\Omega)\, |\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2|\, (f_2'\, f_1' - f_2\, f_1)\, [2 + \ln (f_1' f_2')]\ . \tag{1.11}$$

Finally, adding equations (1.9) and (1.11) together yields:

$$\begin{aligned} 4\frac{dH(t)}{dt} &= \int d^3u_1 \int d^3u_2 \int d\Omega\, \sigma(\Omega)\, |\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2|\, (f_2'\, f_1' - f_2\, f_1) \ln\left(\frac{f_1 f_2}{f_1' f_2'}\right) \\ &= \int d^3u_1 \int d^3u_2 \int d\Omega\, \sigma(\Omega)\, |\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2|\, [f_2'\, f_1'\ (1 - x) \ln x]\ , \end{aligned} \tag{1.12}$$

where we have defined $x \equiv (f_1 f_2)/(f_1' f_2')$. Since $x \geq 0$, the function $(1 - x) \ln x \leq 0$, and therefore we may state the result

$$\frac{dH(t)}{dt} \leq 0\,. \tag{1.13}$$

Equality exists only when $x = 1$, i.e., $f_2'\, f_1' - f_2\, f_1 = 0$ (that is if f is an equilibrium distribution function, f_0), hence proving the Newtonian H-theorem, as required.

Exercise 2

Starting from the (classical) transport equation for the density $n\langle\psi\rangle$, i.e.,

$$\frac{\partial\left(n\langle\psi\rangle\right)}{\partial t}+\frac{\partial\left(n\langle u_i\psi\rangle\right)}{\partial x_i}-n\left\langle u_i\frac{\partial\psi}{\partial x_i}\right\rangle-\frac{n}{m}\left\langle F_i\frac{\partial\psi}{\partial u_i}\right\rangle-\frac{n}{m}\left\langle\frac{\partial F_i}{\partial u_i}\psi\right\rangle=0\,, \tag{2.1}$$

where the quantity ψ is a collisional invariant, show that it is possible to obtain the hydrodynamic equations,

$$\begin{aligned}\frac{\partial\rho}{\partial t}+\frac{\partial\left(\rho v_i\right)}{\partial x_i} &= 0\,, & (2.2)\\ \frac{\partial\left(\rho v_j\right)}{\partial t}+\frac{\partial\left(\rho v_i v_j\right)}{\partial x_i}+\frac{\partial P_{ij}}{\partial x_i}-\frac{\rho}{m}F_j &= 0\,, & (2.3)\\ \frac{\partial\left(\rho\epsilon\right)}{\partial t}+\frac{\partial\left(\rho\epsilon v_i\right)}{\partial x_i}+\frac{\partial q_i}{\partial x_i}+P_{ij}\Lambda^{ij} &= 0\,, & (2.4)\end{aligned}$$

when using the collisional invariants $\psi=m$, $\psi=mu_j$, and $\psi=\frac{1}{2}m|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2$, respectively.

Solution 2

Before proceeding with calculating the moments of the Boltzmann equation, note that we may assume the external force $\vec{\mathbf{F}}$ to be independent of velocity, i.e. $\vec{\mathbf{F}}:=\vec{\mathbf{F}}\left(t,\vec{\mathbf{x}}\right)$. Consequently $\partial F_i/\partial u_i=0$ and we consider the (classical) transport equation as:

$$\frac{\partial\left(n\langle\psi\rangle\right)}{\partial t}+\frac{\partial\left(n\langle u_i\psi\rangle\right)}{\partial x_i}-n\left\langle u_i\frac{\partial\psi}{\partial x_i}\right\rangle-\frac{n}{m}\left\langle F_i\frac{\partial\psi}{\partial u_i}\right\rangle=0\,. \tag{2.5}$$

It is worthwhile to state the following definitions, which will prove necessary to derive the required results:

$$\begin{aligned}P_{ij} &= \rho\langle(u_i-v_i)(u_j-v_j)\rangle\,, & (2.6)\\ \epsilon &= \frac{1}{2}\langle|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2\rangle\,, & (2.7)\\ q_i &= \frac{1}{2}\rho\langle\,(u_i-v_i)\,|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2\rangle\,, & (2.8)\\ \Lambda_{ij} &= \frac{\partial v_j}{\partial x_i}\\ &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\,, & (2.9)\end{aligned}$$

which are respectively, the *pressure tensor*, fluid *specific internal energy*, the *heat flux vector*, and the *strain* of the fluid. The final equality holds because Λ_{ij} is a symmetric tensor.

- **First moment**

  Taking $\psi=m$, the relevant part of the first term in equation (2.5) yields

$$\begin{aligned}n\langle\psi\rangle &= n\langle m\rangle\\ &= n\,m\\ &= \rho\,, & (2.10)\end{aligned}$$

i.e. the fluid density (mass per unit volume). Here n is the number density (number of particles per unit volume) and m is the mass of a fluid particle. The relevant part of the second term yields

$$\begin{aligned} n\langle u_i \psi \rangle &= n\langle u_i m \rangle \\ &= \langle u_i \rangle \\ &= \rho\, v_i\,, \end{aligned} \tag{2.11}$$

where $v_i := \langle u_i \rangle$ is the mean macroscopic velocity (fluid velocity) and indicates a global direction of motion, in contrast to u_i, which the local fluid velocity vector. For the third term we obtain:

$$- n\left\langle u_i \frac{\partial m}{\partial x_i} \right\rangle = 0\,, \tag{2.12}$$

assuming the distribution of matter to be homogeneous. For the fourth term we obtain:

$$- \frac{n}{m}\left\langle F_i \frac{\partial m}{\partial u_i} \right\rangle = 0\,, \tag{2.13}$$

since $\partial m/\partial u_i = 0$. We thus obtain the first hydrodynamic equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial\,(\rho v_i)}{\partial x_i} = 0\,, \tag{2.14}$$

i.e. the *continuity equation.*

- **Second moment**

Taking $\psi = m u_j$, the relevant part of the first term in equation (2.5) yields

$$\begin{aligned} n\langle m u_j \rangle &= nm\langle u_j \rangle \\ &= \rho\, v_j\,. \end{aligned} \tag{2.15}$$

The relevant part of the second term yields

$$\begin{aligned} n\langle u_i m u_j \rangle &= nm\langle u_i u_j \rangle \\ &= \rho\langle u_i u_j \rangle\,. \end{aligned} \tag{2.16}$$

Recall that from the definition of the pressure tensor we may write

$$\begin{aligned} \frac{1}{\rho} P_{ij} &= \langle (u_i - v_i)(u_j - v_j) \rangle \\ &= \langle u_i u_j - u_i v_j - u_j v_i + v_i v_j \rangle \\ &= \langle u_i u_j \rangle - \langle u_i \rangle v_j - \langle u_j \rangle v_i + v_i v_j \\ &= \langle u_i u_j \rangle - v_i v_j - v_i v_j + v_i v_j \\ &= \langle u_i u_j \rangle - v_i v_j\,, \end{aligned} \tag{2.17}$$

from which we may immediately write the relevant part of the second term as

$$n\langle u_i m u_j \rangle = P_{ij} + \rho\, v_i v_j\,. \tag{2.18}$$

For the third term we obtain

$$
\begin{aligned}
-n\left\langle u_i \frac{\partial\left(m u_j\right)}{\partial x_i}\right\rangle &= -\rho\left\langle u_i \frac{\partial u_j}{\partial x_i}\right\rangle \\
&= 0\,, \qquad (2.19)
\end{aligned}
$$

since x_i and u_i are independent canonical variables. For the fourth term we obtain

$$
\begin{aligned}
-\frac{n}{m}\left\langle F_i \frac{\partial\left(m u_j\right)}{\partial u_i}\right\rangle &= -n\left\langle F_i \frac{\partial u_j}{\partial u_i}\right\rangle \\
&= -\frac{\rho}{m}\langle F_i \delta^i{}_j\rangle \\
&= -\frac{\rho}{m}\langle F_j\rangle \\
&= -\frac{\rho}{m} F_j\,, \qquad (2.20)
\end{aligned}
$$

where in the final equality we have assumed that the force is constant. We thus obtain the second hydrodynamic equation

$$
\frac{\partial\left(\rho v_j\right)}{\partial t}+\frac{\partial\left(\rho v_i v_j\right)}{\partial x_i}+\frac{\partial P_{ij}}{\partial x_i}-\frac{\rho}{m} F_j=0\,, \qquad (2.21)
$$

i.e. the *momentum equation*.

- **Third moment**

Taking $\psi=\frac{1}{2} m|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2$, the first term in equation (2.5) yields

$$
\begin{aligned}
\frac{\partial\left(n<\psi>\right)}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{1}{2} \rho\langle|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2\rangle\right] \\
&= \frac{\partial\left(\rho \epsilon\right)}{\partial t}\,. \qquad (2.22)
\end{aligned}
$$

The second term yields

$$
\begin{aligned}
\frac{\partial\left(n<u_i \psi>\right)}{\partial x_i} &= \frac{\partial}{\partial x_i}\left[\frac{1}{2} \rho\langle u_i|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2\rangle\right] \\
&= \frac{\partial}{\partial x_i}\left[\frac{1}{2} \rho\langle\left(u_i-v_i\right)|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2+v_i|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2\rangle\right] \\
&= \frac{\partial}{\partial x_i}\left[\frac{1}{2} \rho\langle\left(u_i-v_i\right)|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2\rangle\right]+\frac{\partial}{\partial x_i}\left[\frac{1}{2} \rho\langle v_i|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2\rangle\right] \\
&= \frac{\partial}{\partial x_i}\left[\frac{1}{2} \rho\langle\left(u_i-v_i\right)|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2\rangle\right]+\frac{\partial}{\partial x_i}\left[\rho\left(\frac{1}{2}\langle|\vec{\mathbf{u}}-\vec{\mathbf{v}}|^2\rangle\right) v_i\right] \\
&= \frac{\partial q_i}{\partial x_i}+\frac{\partial\left(\rho \epsilon v_i\right)}{\partial x_i}\,. \qquad (2.23)
\end{aligned}
$$

Defining $\vec{\mathbf{W}} \equiv \vec{\mathbf{u}} - \vec{\mathbf{v}}$ (i.e. $W_j \equiv u_j - v_j$), the third term yields

$$\begin{aligned} -n\left\langle u_i \frac{\partial \psi}{\partial x_i} \right\rangle &= -\frac{1}{2}\rho\left\langle u_i \frac{\partial}{\partial x_i}\left(\vec{\mathbf{W}} \cdot \vec{\mathbf{W}}\right) \right\rangle \\ &= -\rho\left\langle u_i \left(\vec{\mathbf{W}} \cdot \frac{\partial \vec{\mathbf{W}}}{\partial x_i}\right) \right\rangle \\ &= -\rho\left\langle u_i \, W^j \frac{\partial W_j}{\partial x_i} \right\rangle \\ &= -\rho\left\langle u_i \, \delta^{jk} W_k \frac{\partial W_j}{\partial x_i} \right\rangle . \end{aligned} \tag{2.24}$$

Since δ^{jk} is constant and $\partial W_j / \partial x_i = -\partial v_j / \partial x_i$, the third term may be further simplified as

$$\begin{aligned} -n\left\langle u_i \frac{\partial \psi}{\partial x_i} \right\rangle &= \rho\left\langle u_i \, \delta^{jk} W_k \frac{\partial v_j}{\partial x_i} \right\rangle \\ &= \delta^{jk} \frac{\partial v_j}{\partial x_i} \rho < u_i W_k > \\ &= \delta^{jk} \left(\frac{\partial v_j}{\partial x_i}\right) \left[\rho < u_i \left(u_k - v_k\right) > \right] \\ &= \delta^{jk} \Lambda_{ij} P_{ik} \\ &\equiv \delta_{ij} P_{ik} \Lambda^{jk} \\ &= P_{ij} {\Lambda_i}^j . \end{aligned} \tag{2.25}$$

Note the unusual form of equation (2.25) in terms of its indices. Since $u_i \frac{\partial}{\partial x_i} \equiv \vec{\mathbf{u}} \cdot \vec{\nabla}$ is a divergence, the summation over index "i" implies the evaluation of $\vec{\mathbf{u}} \cdot \vec{\nabla}$, whereas the summation over "k" is a summation in the "Einstein summation convention" sense. Strictly speaking, this term should be written

$$\begin{aligned} P_{ij} {\Lambda_i}^j &= \sum_{i=1}^{3} P_{ij} {\Lambda_i}^j \\ &= P_{1j} {\Lambda_1}^j + P_{2j} {\Lambda_2}^j + P_{3j} {\Lambda_3}^j . \end{aligned} \tag{2.26}$$

This issue is to do with the definition of contravariant (index up) and covariant (index down) vectors. Since the sum is implied, and we do not concern ourselves at this point with covariance, without loss of generality we may write

$$-n\left\langle u_i \frac{\partial \psi}{\partial x_i} \right\rangle = P_{ij} \Lambda^{ij} . \tag{2.27}$$

We thus obtain the third hydrodynamic equation

$$\frac{\partial (\rho \epsilon)}{\partial t} + \frac{\partial (\rho \epsilon v_i)}{\partial x_i} + \frac{\partial q_i}{\partial x_i} + P_{ij} \Lambda^{ij} = 0 , \tag{2.28}$$

i.e. the *energy equation*.

Exercise 3

Optional. Show that equations (2.2)–(2.4) may also be written as:

$$\frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i} + \frac{1}{\rho}\frac{\partial P_{ij}}{\partial x_i} - \frac{1}{m} F_j = 0\,, \tag{3.1}$$

$$\frac{\partial \epsilon}{\partial t} + v_i \frac{\partial \epsilon}{\partial x_i} + \frac{1}{\rho}\frac{\partial q_i}{\partial x_i} + \frac{1}{\rho} P_{ij}\Lambda^{ij} = 0\,. \tag{3.2}$$

Solution 3

For the first equation, consider the expansion of the derivatives of the first two terms of equation (2.3)

$$\begin{aligned}
\frac{\partial(\rho v_j)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_i} &= \rho\frac{\partial v_j}{\partial t} + v_j\frac{\partial \rho}{\partial t} + (\rho v_i)\frac{\partial v_j}{\partial x_i} + v_j \frac{\partial(\rho v_i)}{\partial x_i} \\
&= v_j \cancelto{0}{\left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i}\right]} + \rho\left(\frac{\partial v_j}{\partial t} + v_i\frac{\partial v_j}{\partial x_i}\right) \\
&= \rho\left(\frac{\partial v_j}{\partial t} + v_i\frac{\partial v_j}{\partial x_i}\right)\,,
\end{aligned} \tag{3.3}$$

where on the second line we used the continuity equation to remove the term in square brackets. Therefore equation (2.3) becomes

$$\rho\left(\frac{\partial v_j}{\partial t} + v_i\frac{\partial v_j}{\partial x_i}\right) + \frac{\partial P_{ij}}{\partial x_i} - \frac{\rho}{m}F_j = 0\,, \tag{3.4}$$

which upon dividing throughout by ρ yields

$$\frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i} + \frac{1}{\rho}\frac{\partial P_{ij}}{\partial x_i} - \frac{1}{m} F_j = 0\,, \tag{3.5}$$

as required. Similarly, we consider the expansion of the derivatives of the first two terms of equation (2.4)

$$\begin{aligned}
\frac{\partial(\rho \epsilon)}{\partial t} + \frac{\partial(\rho \epsilon v_i)}{\partial x_i} &= \epsilon\frac{\partial \rho}{\partial t} + \rho\frac{\partial \epsilon}{\partial t} + (\rho v_i)\frac{\partial \epsilon}{\partial x_i} + \epsilon \frac{\partial(\rho v_i)}{\partial x_i} \\
&= \epsilon \cancelto{0}{\left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i}\right]} + \rho\left(\frac{\partial \epsilon}{\partial t} + v_i\frac{\partial \epsilon}{\partial x_i}\right) \\
&= \rho\left(\frac{\partial \epsilon}{\partial t} + v_i\frac{\partial \epsilon}{\partial x_i}\right)\,.
\end{aligned} \tag{3.6}$$

Therefore equation (2.4) becomes

$$\rho\left(\frac{\partial \epsilon}{\partial t} + v_i\frac{\partial \epsilon}{\partial x_i}\right) + \frac{\partial q_i}{\partial x_i} + P_{ij}\Lambda^{ij} = 0\,, \tag{3.7}$$

which upon dividing throughout by ρ yields

$$\frac{\partial \epsilon}{\partial t} + v_i \frac{\partial \epsilon}{\partial x_i} + \frac{1}{\rho}\frac{\partial q_i}{\partial x_i} + \frac{1}{\rho} P_{ij}\Lambda^{ij} = 0\,, \tag{3.8}$$

as required.