General Relativity: Solutions to exercises in Lecture V

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Exercise 1

Let **F** be a rank-2 antisymmetric tensor, **G** a rank-2 symmetric tensor and **X** and rank-3 antisymmetric tensor. Provide explicit expressions for the following tensors: $F_{\mu\nu}$, $F_{[\mu\nu]}$, $F_{(\mu\nu)}$, $G_{[\mu\nu]}$, $G_{(\mu\nu)}$, $X_{[\alpha\beta\gamma]}$, $X_{(\alpha\beta\gamma)}$, $X_{[\alpha\beta]\gamma}$, $X_{(\alpha\beta)\gamma}$, $X_{[\alpha\beta](\gamma)}$ and $X_{(\alpha\beta)[\gamma]}$.

Solution 1

- $F_{\mu\nu} = -F_{\nu\mu}$
- $F_{[\mu\nu]} = \frac{1}{2}\left(F_{\mu\nu} - F_{\nu\mu}\right) = \frac{1}{2}\left(F_{\mu\nu} + F_{\mu\nu}\right) = F_{\mu\nu}$
- $F_{(\mu\nu)} = \frac{1}{2}\left(F_{\mu\nu} + F_{\nu\mu}\right) = \frac{1}{2}\left(F_{\mu\nu} - F_{\mu\nu}\right) = 0$
- $G_{[\mu\nu]} = 0$ (the antisymmetric part of a totally symmetric tensor must be zero)
- $G_{(\mu\nu)} = G_{\mu\nu}$
- $X_{[\alpha\beta\gamma]} = \frac{1}{3!}\left(X_{\alpha\beta\gamma} - X_{\beta\alpha\gamma} + X_{\gamma\alpha\beta} - X_{\alpha\gamma\beta} + X_{\beta\gamma\alpha} - X_{\gamma\beta\alpha}\right) = \frac{1}{6}\left(2X_{\alpha\beta\gamma} + 2X_{\gamma\alpha\beta} + 2X_{\beta\gamma\alpha}\right)$

  $= \frac{1}{3}\left(X_{\alpha\beta\gamma} + X_{\gamma\alpha\beta} + X_{\beta\gamma\alpha}\right)$
- $X_{(\alpha\beta\gamma)} = \frac{1}{3!}\left(X_{\alpha\beta\gamma} + X_{\beta\alpha\gamma} + X_{\gamma\alpha\beta} + X_{\alpha\gamma\beta} + X_{\beta\gamma\alpha} + X_{\gamma\beta\alpha}\right)$

  $= \frac{1}{3!}\left(X_{\alpha\beta\gamma} - X_{\alpha\beta\gamma} + X_{\gamma\alpha\beta} - X_{\gamma\alpha\beta} + X_{\beta\gamma\alpha} - X_{\beta\gamma\alpha}\right) = 0$
- $X_{[\alpha\beta]\gamma} = \frac{1}{2}\left(X_{\alpha\beta\gamma} - X_{\beta\alpha\gamma}\right) = X_{\alpha\beta\gamma}$
- $X_{(\alpha\beta)\gamma} = \frac{1}{2}\left(X_{\alpha\beta\gamma} + X_{\beta\alpha\gamma}\right) = 0$
- $X_{[\alpha\beta](\gamma)} = X_{[\alpha\beta]\gamma} = X_{\alpha\beta\gamma}$
- $X_{(\alpha\beta)[\gamma]} = X_{(\alpha\beta)\gamma} = 0$

Exercise 2

Prove the following identities:

- $X_{((\alpha_1\ \alpha_2 \dots \alpha_n))} = X_{(\alpha_1\ \alpha_2 \dots \alpha_n)}$
- $X_{[[\alpha_1\ \alpha_2 \dots \alpha_n]]} = X_{[\alpha_1\ \alpha_2 \dots \alpha_n]}$
- $X_{(\alpha_1 \dots [\alpha_l\ \alpha_m] \dots \alpha_n)} = 0$
- $X_{[\alpha_1 \dots [\alpha_l\ \alpha_m] \dots \alpha_n]} = X_{[\alpha_1 \dots \alpha_l\ \alpha_m \dots \alpha_n]}$

Solution 2

- If $Y_{\alpha_1 \alpha_2 \dots \alpha_n}$ is a totally symmetric tensor then we may write

  $$Y_{\alpha_{\pi_1} \alpha_{\pi_2} \dots \alpha_{\pi_n}} = Y_{\alpha_1 \alpha_2 \dots \alpha_n} \ , \tag{1}$$

  where π_i denotes permutation over the index i. We may thus write the symmetric part of $\mathbf{Y}$ as

  $$\begin{aligned} Y_{(\alpha_1 \alpha_2 \dots \alpha_n)} &= \frac{1}{n!} \sum Y_{\alpha_{\pi_1} \alpha_{\pi_2} \dots \alpha_{\pi_n}} \\ &= Y_{\alpha_1 \alpha_2 \dots \alpha_n} \ . \end{aligned} \tag{2}$$

  Now, letting $X_{(\alpha_1 \alpha_2 \dots \alpha_n)} = Y_{\alpha_1 \alpha_2 \dots \alpha_n}$ we may write

  $$\begin{aligned} Y_{(\alpha_1 \alpha_2 \dots \alpha_n)} &= X_{((\alpha_1 \alpha_2 \dots \alpha_n))} \\ &= X_{(\alpha_1 \alpha_2 \dots \alpha_n)} \ , \end{aligned} \tag{3}$$

  as required.

- Similarly to the previous question, if $Y_{\alpha_1 \alpha_2 \dots \alpha_n}$ is a totally antisymmetric tensor then we may write

  $$(-1)^\pi Y_{\alpha_{\pi_1} \alpha_{\pi_2} \dots \alpha_{\pi_n}} = Y_{\alpha_1 \alpha_2 \dots \alpha_n} \ , \tag{4}$$

  We may thus write the antisymmetric part of $\mathbf{Y}$ as

  $$\begin{aligned} Y_{[\alpha_1 \alpha_2 \dots \alpha_n]} &= \frac{1}{n!} \sum (-1)^\pi \, Y_{\alpha_{\pi_1} \alpha_{\pi_2} \dots \alpha_{\pi_n}} \\ &= Y_{\alpha_1 \alpha_2 \dots \alpha_n} \ . \end{aligned} \tag{5}$$

  Now, letting $X_{[\alpha_1 \alpha_2 \dots \alpha_n]} = Y_{\alpha_1 \alpha_2 \dots \alpha_n}$ we may write

  $$\begin{aligned} Y_{[\alpha_1 \alpha_2 \dots \alpha_n]} &= X_{[[\alpha_1 \alpha_2 \dots \alpha_n]]} \\ &= X_{[\alpha_1 \alpha_2 \dots \alpha_n]} \ , \end{aligned} \tag{6}$$

  as required.

- By symmetry (outer round brackets) we have

$$X_{(\alpha_1\ldots[\alpha_l\alpha_m]\ldots\alpha_n)} = X_{(\alpha_1\ldots[\alpha_m\alpha_l]\ldots\alpha_n)} \ , \tag{7}$$

but by antisymmetry (inner square brackets) we have

$$X_{(\alpha_1\ldots[\alpha_l\alpha_m]\ldots\alpha_n)} = -X_{(\alpha_1\ldots[\alpha_m\alpha_l]\ldots\alpha_n)} \ , \tag{8}$$

and thus we may conclude that

$$X_{(\alpha_1\ldots[\alpha_l\alpha_m]\ldots\alpha_n)} = 0 \ , \tag{9}$$

as required.

- First consider

$$X_{\alpha_1\ldots[\alpha_l\alpha_m]\ldots\alpha_n} = \frac{1}{2}\left(X_{\alpha_1\ldots\alpha_l\alpha_m\ldots\alpha_n} - X_{\alpha_1\ldots\alpha_m\alpha_l\ldots\alpha_n}\right) \ . \tag{10}$$

Now take the full antisymmetric part of this

$$\begin{aligned} X_{[\alpha_1\ldots[\alpha_l\alpha_m]\ldots\alpha_n]} &= \frac{1}{2n!}\sum(-1)^\pi \left(X_{\alpha_{\pi_1}\ldots\alpha_{\pi_l}\alpha_{\pi_m}\ldots\alpha_{\pi_n}} - X_{\alpha_{\pi_1}\ldots\alpha_{\pi_m}\alpha_{\pi_l}\ldots\alpha_{\pi_n}}\right) \\ &= \frac{1}{n!}\sum(-1)^\pi X_{\alpha_{\pi_1}\ldots\alpha_{\pi_l}\alpha_{\pi_m}\ldots\alpha_{\pi_n}} \\ &= X_{[\alpha_1\ldots\alpha_l\alpha_m\ldots\alpha_n]} \ , \end{aligned} \tag{11}$$

where we have used the fact that $X_{\alpha_{\pi_1}\ldots\alpha_{\pi_m}\alpha_{\pi_l}\ldots\alpha_{\pi_n}} = -X_{\alpha_{\pi_1}\ldots\alpha_{\pi_l}\alpha_{\pi_m}\ldots\alpha_{\pi_n}}$, as required.

Exercise 3

Let $\mathbf{F}$ be a rank-2 antisymmetric tensor with components $F^{\mu\nu}$. From $\mathbf{F}$ construct another rank-2 tensor antisymmetric tensor $^*\mathbf{F}$ such that

$$^*\mathbf{F} := \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}\,\mathbf{e}_\mu \otimes \mathbf{e}_\nu \ . \tag{12}$$

The tensor $^*\mathbf{F}$ is usually referred to as the *dual* of $\mathbf{F}$. Show that the following is true

$$^*\left(^*\mathbf{F}\right) = -\mathbf{F} \ . \tag{13}$$

Solution 3

We may write the dual of $\mathbf{F}$ in contravariant index form as

$$^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta} \ . \tag{14}$$

Accordingly, the covariant form may be written, using the relation $F_{\mu\nu} = g_{\mu\gamma}g_{\nu\delta}F^{\gamma\delta}$, as

$$\begin{aligned} ^*F_{\mu\nu} &= \frac{1}{2}g_{\mu\gamma}g_{\nu\delta}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta} \\ &= \frac{1}{2}\epsilon^{\alpha\beta}{}_{\mu\nu}F_{\alpha\beta} \\ &= \frac{1}{2}g^{\alpha\gamma}g^{\beta\delta}\epsilon_{\gamma\delta\mu\nu}F_{\alpha\beta} \\ &= \frac{1}{2}\epsilon_{\gamma\delta\mu\nu}F^{\gamma\delta} \ . \end{aligned} \tag{15}$$

We may now write

$$\begin{aligned}
{}^*\left({}^*F^{\mu\nu}\right) &= \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\left({}^*F_{\alpha\beta}\right) \\
&= \frac{1}{4}\epsilon^{\alpha\beta\mu\nu}\epsilon_{\gamma\delta\alpha\beta}F^{\gamma\delta} \\
&= \frac{1}{4}\left(-2!\,\delta^{\mu\nu}_{\gamma\delta}\right)F^{\gamma\delta} \\
&= -\frac{1}{2}\left(\delta^{\mu}_{\gamma}\,\delta^{\nu}_{\delta} - \delta^{\mu}_{\delta}\,\delta^{\nu}_{\gamma}\right)F^{\gamma\delta} \\
&= -\frac{1}{2}\left(F^{\mu\nu} - F^{\nu\mu}\right) \\
&= -F^{\mu\nu}\ . \qquad (16)
\end{aligned}$$

Thus we obtain

$$ {}^*\left({}^*\mathbf{F}\right) = -\mathbf{F}\ , \qquad (17)$$

as required.

Exercise 4

Let $\mathbf{V}$ be a rank-3 tensor with components $V^{\alpha\beta\gamma}$ and define

$$\left({}^*V\right)^{\alpha\beta\gamma} := V_{\mu}\epsilon^{\mu\alpha\beta\gamma}\ . \qquad (18)$$

Show that the following is true

$$V^{\mu}V_{\mu} = -\frac{1}{3!}\left({}^*V\right)^{\alpha\beta\gamma}\left({}^*V\right)_{\alpha\beta\gamma}\ . \qquad (19)$$

Solution 4

In addition to equation (18), for fully covariant $\mathbf{V}$ we may write

$$\left({}^*V\right)_{\alpha\beta\gamma} = V^{\nu}\epsilon_{\nu\alpha\beta\gamma}\ . \qquad (20)$$

From this we may immediately calculate the contraction as

$$\begin{aligned}
\left({}^*V\right)^{\alpha\beta\gamma}\left({}^*V\right)_{\alpha\beta\gamma} &= V^{\nu}V_{\mu}\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\nu\alpha\beta\gamma} \\
&= V^{\nu}V_{\mu}\left(-3!\,\delta^{\mu}_{\nu}\right) \\
&= -3!\,V^{\mu}V_{\mu}\ , \qquad (21)
\end{aligned}$$

and hence we obtain

$$V^{\mu}V_{\mu} = -\frac{1}{3!}\left({}^*V\right)^{\alpha\beta\gamma}\left({}^*V\right)_{\alpha\beta\gamma}\ , \qquad (22)$$

as required.