General Relativity: Solutions to exercises in Lecture III

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Exercise 1

Consider $\boldsymbol{T}$ as a contravariant tensor of rank 2 with components $T^{\mu\nu}$. Under what conditions can this tensor be cast as the product of two contravariant vectors $\mathbf{U}$ and $\mathbf{V}$, i.e. such that $T^{\mu\nu} = U^{\mu}V^{\nu}$?

Solution 1

In a given basis $\boldsymbol{T}$ is represented by a matrix $T^{\mu\nu}$. In these terms a necessary and sufficient condition to enable $T^{\mu\nu}$ to be written as $T^{\mu\nu} = U^{\mu}V^{\nu}$ is that all columns of the matrix $T^{\mu\nu}$ must be proportional to each other (linearly dependent). As an example, consider the following matrix:

$$T^{\mu\nu} = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 8 & 16 \\ 3 & 6 & 12 & 24 \\ 4 & 8 & 16 & 32 \end{pmatrix} .$$

Since the columns of this matrix are proportional to one another, we may choose $U^{\mu} = (1,2,3,4)$ and $V^{\nu} = (1,2,4,8)$, thus satisfying $T^{\mu\nu} = U^{\mu}V^{\nu}$.

Let us now consider this in a co-ordinate independent (covariant) way. $T^{\mu\nu} = U^{\mu}V^{\nu}$ if and only if $S^{\mu} = T^{\mu\nu}x_{\nu}$ is in the same direction, for any given x_{ν}.

Consider the set of orthonormal basis vectors $\mathbf{e}^0$, $\mathbf{e}^1$, $\mathbf{e}^2$ and $\mathbf{e}^3$ which by definition must satisfy $\mathbf{e}^{\mu}\mathbf{e}_{\nu} = \delta^{\mu}{}_{\nu}$. The direction of S^{μ} is independent of the choice of x_{ν} (by linearity) if and only if it is independent of our basis vectors $\mathbf{e}^0$, $\mathbf{e}^1$, $\mathbf{e}^2$ and $\mathbf{e}^3$. As such we may obtain the following condition:

$$\begin{aligned} T^{\mu\nu}e^{\alpha}{}_{\nu} &= T^{\mu\alpha} \\ &= \mathcal{C}_{\alpha}S^{\mu} , \end{aligned}$$

where $\mathcal{C}_{\alpha} = (\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ are constants. Explicitly:

$$\begin{aligned} T^{\mu 0} &= \mathcal{C}_0 S^{\mu} , \\ T^{\mu 1} &= \mathcal{C}_1 S^{\mu} , \\ T^{\mu 2} &= \mathcal{C}_2 S^{\mu} , \\ T^{\mu 3} &= \mathcal{C}_3 S^{\mu} . \end{aligned}$$

Thus the columns must be proportional to each other.

Exercise 2

Consider the following equation:

$$T^{\mu\nu} = U^\mu + V^\nu \ .$$

Is $\boldsymbol{T}$ a generic tensor?

Solution 2

$\boldsymbol{T}$ is not a generic tensor. If $\boldsymbol{T}$ were a tensor then $T^{\mu\nu} A_\mu B_\nu$ would have to be a scalar. Instead, one obtains

$$\begin{aligned} T^{\mu\nu} A_\mu B_\nu &= (U^\mu + V^\nu)\, A_\mu B_\nu \\ &= (U^\mu A_\mu)\, B_\nu + (V^\nu B_\nu)\, A_\mu \\ &= \alpha B_\nu + \beta A_\mu \ , \end{aligned}$$

where $\alpha \equiv U^\mu A_\mu$ and $\beta \equiv V^\nu B_\nu$ are both scalars. It immediately follows that $\alpha B_\nu + \beta A_\mu$ is not a scalar and therefore $\boldsymbol{T}$ is not a generic tensor.

Exercise 3

Consider $\boldsymbol{F}$ as a tensor of rank 2 with covariant components $F_{\mu\nu}$ and that is also antisymmetric in one co-ordinate system, i.e. $F_{\mu\nu} = -F_{\nu\mu}$.

- Show that $F_{\mu\nu}$ is antisymmetric in all co-ordinate systems.
- Does the antisymmetry in the covariant indices also apply to the contravariant indices?
- If so, show that $F^{\mu\nu}$ is antisymmetric in all co-ordinate systems.

Solution 3

First consider the transformation of $F_{\mu\nu}$ into another co-ordinate system:

$$\begin{aligned} F_{\mu'\nu'} &= \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} F_{\mu\nu} \\ &= -\Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} F_{\nu\mu} \\ &= -\Lambda^\nu{}_{\mu'} \Lambda^\mu{}_{\nu'} F_{\mu\nu} \\ &= -F_{\nu'\mu'} \ . \end{aligned}$$

It immediately follows that $F_{\mu\nu}$ is symmetric in all co-ordinate systems. The antisymmetry in covariant indices indeed also applies to the contravariant indices since $\boldsymbol{F}$ is a tensor. This can be shown by considering the following:

$$\begin{aligned} F^{\mu\nu} &= g^{\mu\mu'} g^{\nu\nu'} F_{\mu'\nu'} \\ &= -g^{\mu\mu'} g^{\nu\nu'} F_{\nu'\mu'} \\ &= -g^{\mu\nu'} g^{\nu\mu'} F_{\mu'\nu'} \\ &= -F^{\nu\mu} \ , \end{aligned}$$

as required.

Exercise 4

For the first part of the question, consider the antisymmetric tensor $A_{\mu\nu}$ such that $A_{\mu\nu} = -A_{\nu\mu}$ and the symmetric tensor $B^{\mu\nu}$ such that $B^{\mu\nu} = B^{\nu\mu}$. Prove the following identities:

$$A_{\mu\nu} B^{\mu\nu} = 0 \,, \tag{1}$$

$$V^{\mu\nu} A_{\mu\nu} = \frac{1}{2}\left(V^{\mu\nu} - V^{\nu\mu}\right) A_{\mu\nu} \,, \tag{2}$$

$$V^{\mu\nu} B_{\mu\nu} = \frac{1}{2}\left(V^{\mu\nu} + V^{\nu\mu}\right) B_{\mu\nu} \,. \tag{3}$$

Solution 4

For the first identity consider the following:

$$\begin{aligned} A_{\mu\nu} B^{\mu\nu} &= -A_{\nu\mu} B^{\mu\nu} \\ &= -A_{\mu\nu} B^{\mu\nu} \\ &= 0 \,, \end{aligned}$$

where we have used the antisymmetry of $A_{\mu\nu}$ in the first step and relabelling dummy indices and the symmetry of $B^{\mu\nu}$ in the second step, hence the required result. For the second and third parts, recall that a generic rank 2 tensor may be written in terms of a symmetric and antisymmetric component as follows:

$$\begin{aligned} V^{\mu\nu} &= \frac{1}{2}\left(V^{\mu\nu} + V^{\nu\mu}\right) + \frac{1}{2}\left(V^{\mu\nu} - V^{\nu\mu}\right) \\ &= V^{(\mu\nu)} + V^{[\mu\nu]} \,. \end{aligned}$$

Now consider the action of the antisymmetric tensor $A_{\mu\nu}$ on the symmetric part of $V^{\mu\nu}$, i.e. $V^{(\mu\nu)}$:

$$\begin{aligned} V^{(\mu\nu)} A_{\mu\nu} &= \frac{1}{2}\left(V^{\mu\nu} A_{\mu\nu} + V^{\nu\mu} A_{\mu\nu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu} A_{\mu\nu} + V^{\mu\nu} A_{\nu\mu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu} A_{\mu\nu} - V^{\mu\nu} A_{\mu\nu}\right) \\ &= 0 \,, \end{aligned}$$

where in the first step we have relabelled dummy indices in the second term, and in the second step we have used the antisymmetry of $A_{\mu\nu}$. In a similar fashion we may also consider the action of the symmetric tensor $B_{\mu\nu}$ on the antisymmetric part of $V^{\mu\nu}$, i.e. $V^{[\mu\nu]}$:

$$\begin{aligned} V^{[\mu\nu]} B_{\mu\nu} &= \frac{1}{2}\left(V^{\mu\nu} B_{\mu\nu} - V^{\nu\mu} B_{\mu\nu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu} B_{\mu\nu} - V^{\mu\nu} B_{\nu\mu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu} B_{\mu\nu} - V^{\mu\nu} B_{\mu\nu}\right) \\ &= 0 \,. \end{aligned}$$

We are now in a position to tackle the second and third identities. For the second identity, consider the following:

$$\begin{aligned} V^{\mu\nu}A_{\mu\nu} &= V^{(\mu\nu)}A_{\mu\nu} + V^{[\mu\nu]}A_{\mu\nu} \\ &= V^{[\mu\nu]}A_{\mu\nu} \\ &= \frac{1}{2}\left(V^{\mu\nu} - V^{\nu\mu}\right)A_{\mu\nu} \ , \end{aligned}$$

as required.

Finally, we consider the third identity:

$$\begin{aligned} V^{\mu\nu}B_{\mu\nu} &= V^{(\mu\nu)}B_{\mu\nu} + V^{[\mu\nu]}B_{\mu\nu} \\ &= V^{(\mu\nu)}B_{\mu\nu} \\ &= \frac{1}{2}\left(V^{\mu\nu} + V^{\nu\mu}\right)B_{\mu\nu} \ , \end{aligned}$$

as required.