General Relativity: Solutions to exercises in Lecture II

Ziri Younsi

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Exercise 1

Consider two co-ordinate systems in a two dimensional space $\{x^{\mu}\} = (x, y)$ and $\{x^{\mu'}\} = (r, \theta)$ which are related through the well-known co-ordinate transformation

$$\boldsymbol{f} : \begin{cases} r = \left(x^2 + y^2\right)^{1/2} \\ \theta = \arctan\left(y/x\right) \end{cases}$$

and its inverse

$$\boldsymbol{f}^{-1} : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Discuss the differences between the transformation matrix employed to transform a covector in this space

$$\left(\tilde{\mathrm{d}}x\right)_{\mu} = \Lambda^{\mu'}{}_{\mu}\left(\tilde{\mathrm{d}}x\right)_{\mu'} \ , \tag{1}$$

and the one employed in the co-ordinate transformation

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu}\, x^{\mu} \ . \tag{2}$$

Solution 1

The matrix involved in the transformation of the gradient $\left(\tilde{\mathrm{d}}x\right)_{\mu} = \Lambda^{\mu'}{}_{\mu}\left(\tilde{\mathrm{d}}x\right)_{\mu'}$ is different from the matrix used in the transformation $x^{\mu'} = \Lambda^{\mu'}{}_{\mu}\, x^{\mu}$. The two matrices, although written identically, are in fact transposes of each other.

To illustrate this, consider the co-ordinate systems $\{x^{\mu}\} = (x, y)$ and $\{x^{\mu'}\} = (r, \theta)$. It follows that $x^1 = x$, $x^2 = y$; $x^{1'} = r$, $x^{2'} = \theta$. One may now calculate the transformation between co-ordinate systems as:

$$\begin{aligned} x^{1'} &= r = \Lambda^{1'}{}_{\mu}\, x^{\mu} \\ &= \Lambda^{1'}{}_{1}\, x^{1} + \Lambda^{2'}{}_{2}\, x^{2} \\ &= \frac{\partial x^{1'}}{\partial x^{1}}\, x^{1} + \frac{\partial x^{1'}}{\partial x^{2}}\, x^{2} \\ &= \frac{\partial r}{\partial x}\, x + \frac{\partial r}{\partial y}\, y \ , \end{aligned} \tag{3}$$

and similarly $x^{2'} = (\partial\theta/\partial x)x + (\partial\theta/\partial y)y$. We may now write the transformation matrix as:

$$\Lambda^{\mu'}_{\ \ \mu} = \begin{pmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{pmatrix} \ . \tag{4}$$

On the other hand, for $\left(\tilde{\mathrm{d}}x\right)_\mu = \Lambda^{\mu'}_{\ \ \mu}\left(\tilde{\mathrm{d}}x\right)_{\mu'}$, consider the following explicit transformation:

$$\begin{aligned} \left(\tilde{\mathrm{d}}x\right)_1 &= \Lambda^{\mu'}_{\ \ 1}\left(\tilde{\mathrm{d}}x\right)_{\mu'} \\ &= \Lambda^{1'}_{\ \ 1}\left(\tilde{\mathrm{d}}x\right)_{1'} + \Lambda^{2'}_{\ \ 1}\left(\tilde{\mathrm{d}}x\right)_{2'} \\ &= \frac{\partial r}{\partial x}\left(\tilde{\mathrm{d}}x\right)_{1'} + \frac{\partial\theta}{\partial x}\left(\tilde{\mathrm{d}}x\right)_{2'} \ . \end{aligned} \tag{5}$$

Similarly, one finds $\left(\tilde{\mathrm{d}}x\right)_2 = (\partial r/\partial y)\left(\tilde{\mathrm{d}}x\right)_{1'} + (\partial\theta/\partial y)\left(\tilde{\mathrm{d}}x\right)_{2'}$. The transformation matrix may now be written as:

$$\begin{aligned} \Lambda^{\mu'}_{\ \ \mu} &= \begin{pmatrix} \partial r/\partial x & \partial\theta/\partial x \\ \partial r/\partial y & \partial\theta/\partial y \end{pmatrix} && (6) \\ &= \chi^{\mu'}_{\ \ \mu} \ . && (7) \end{aligned}$$

Clearly $\left(\chi^{\mu'}_{\ \ \mu}\right)^{\mathrm{T}} = \Lambda^{\mu'}_{\ \ \mu}$ from equation (4), i.e. the transformation matrices are transposes of each other, as required.

Exercise 2

Consider two co-ordinate systems in a four-dimensional spacetime $x^\mu = (t, x, y, z)$ and $x^{\mu'} = (u, v, y, z)$ that are related through the co-ordinate transformation

$$\boldsymbol{f} : \begin{cases} u = t - x \\ v = t + x \end{cases}$$

and its inverse

$$\boldsymbol{f}^{-1} : \begin{cases} t = \frac{1}{2}(v + u) \\ x = \frac{1}{2}(v - u) \end{cases}$$

- Compute the matrices employed in the transformations

$$x^{\mu'} = \Lambda^{\mu'}_{\ \ \mu}\, x^\mu \qquad\qquad x^\mu = \Lambda^{\mu}_{\ \ \mu'}\, x^{\mu'} \ . \tag{8}$$

- Consider a four-vector with components $U^\mu = (1, 0, 0, 0)^{\mathrm{T}}$ in the co-ordinate system x^μ and compute the new components $U^{\mu'}$ in the co-ordinate system $x^{\mu'}$.
- Repeat the calculation for the new vector $V^\mu = (-1/2, 1/2, 0, 0)^{\mathrm{T}}$. Interpret the results.

Solution 2

For the first part of the question, computing the transformation matrices, first consider $\Lambda^{\mu'}{}_{\mu}$.

$$\Lambda^{0'}_{\mu} = \frac{\partial x^{0'}}{\partial x^{\mu}} = \frac{\partial u}{\partial x^{\mu}} , \tag{9}$$

from which one obtains the following non-zero components:

$$\Lambda^{0'}{}_{0} = \frac{\partial u}{\partial t} = 1 , \tag{10}$$

$$\Lambda^{0'}{}_{1} = \frac{\partial u}{\partial x} = -1 . \tag{11}$$

Similarly,

$$\Lambda^{1'}{}_{\mu} = \frac{\partial x^{1'}}{\partial x^{\mu}} = \frac{\partial v}{\partial x^{\mu}} , \tag{12}$$

from which one obtains the following non-zero components:

$$\Lambda^{1'}{}_{0} = \frac{\partial v}{\partial t} = 1 , \tag{13}$$

$$\Lambda^{1'}{}_{1} = \frac{\partial v}{\partial x} = 1 . \tag{14}$$

Finally, one may also show that the remaining non-zero components of $\Lambda^{\mu'}{}_{\mu}$ are

$$\Lambda^{2'}{}_{2} = 1 , \tag{15}$$

$$\Lambda^{3'}{}_{3} = 1 . \tag{16}$$

The transformation matrix may now be written as

$$\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{17}$$

For the inverse transformation matrix we follow the same procedure, from which the inverse transformation matrix is found as

$$\Lambda^{\mu}{}_{\mu'} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{18}$$

The second part of the question asks to calculate U^μ in the new co-ordinate system, i.e. $U^{\mu'}$. Whilst it is obvious that one can do this through matrix multiplication, consider instead the following:

$$\begin{aligned} U^{\mu'} &= \Lambda^{\mu'}_{\ \mu} U^\mu \\ &= \Lambda^{\mu'}_{\ 0} U^0 \\ &= \Lambda^{\mu'}_{\ 0} , \end{aligned} \tag{19}$$

where the fact that the only non-zero component of U^μ is U^0 has been used. One can then read directly from equation (17) the solution as

$$U^\mu = (1, 1, 0, 0)^{\mathrm{T}} . \tag{20}$$

For the third and final part of this question one can again apply matrix multiplication to obtain the result, or consider the basis components as follows:

$$\begin{aligned} V^{\mu'} &= \Lambda^{\mu'}_{\ \mu} V^\mu \\ &= \Lambda^{\mu'}_{\ 0} V^0 + \Lambda^{\mu'}_{\ 1} V^1 . \end{aligned} \tag{21}$$

Considering this term by term yields

$$\begin{aligned} V^{0'} &= \Lambda^{0'}_{\ 0} V^0 + \Lambda^{0'}_{\ 1} V^1 \\ &= (1).(-1/2) + (-1).(1/2) \\ &= -1 , \end{aligned} \tag{22}$$

and

$$\begin{aligned} V^{1'} &= \Lambda^{1'}_{\ 0} V^0 + \Lambda^{1'}_{\ 1} V^1 \\ &= (1).(-1/2) + (1).(1/2) \\ &= 0 , \end{aligned} \tag{23}$$

from which it immediately follows that

$$V^{\mu'} = (-1, 0, 0, 0)^{\mathrm{T}} . \tag{24}$$

The second part may be interpreted as follows. In $\{x^\mu\}$ the four-vector U^μ represents a particle at rest, since all spatial components are zero: the particle may be represented as a vertical worldline in a $1 + 1$-spacetime. However, when transforming to $\{x^{\mu'}\}$ one finds that $U^{\mu'}$ has two non-zero components, implying that the particle no longer appears stationary and is moving with a constant velocity. Represented as a worldline in a $1 + 1$-spacetime (u, v) the worldline would be a line of constant positive (and finite) gradient.

For the third part, the vector V^μ has non-zero spatial components and so has a velocity of -1 in the x-direction. Represented as a worldline in a $1 + 1$-spacetime (t, x) it would be represented by a line of constant, finite and non-zero gradient. However, when transformed into $\{x^{\mu'}\}$, the four-vector $V^{\mu'}$ has zero spatial components. So in the co-ordinate system $\{x^{\mu'}\}$ the four-vector V^μ appears stationary.

Exercise 3

Consider a 1 + 1 representation of the sub-spaces with two co-ordinate systems (t, x) and (u, v).

- Draw in the two spacetimes the worldline of a particle with velocity $\dot{x} := \mathrm{d}x/\mathrm{d}t = 0$.
- Draw in the two spacetimes the worldline of a particle with velocity $\dot{x} := k$ $(x = kt)$ with $k < 1$.
- Interpret the results.

Solution 3

In this question it is assumed we use the co-ordinate transformations as defined in Exercise 2.

For the first part, let us term the first particle as particle A. Since $\dot{x}_A = 0$ this implies $x_A = \text{const}$. The particle is stationary and at rest in the (t, x) co-ordinate system. In the (u, v) co-ordinate system one may write

$$u_A = t - x_A \; , \tag{25}$$
$$v_A = t + x_A \; , \tag{26}$$

from which one may conclude

$$\frac{u_A}{v_A} = \frac{t - x_A}{t + x_A} < 1. \tag{27}$$

Since $\partial u_A/\partial v_A \simeq (t - x_A)/(t + x_A) = (v_A - 2x_A)/v$. Integrating this yields

$$u(v) = v - \frac{2x_A}{v^2} + \text{const.} \tag{28}$$

We may set the integration constant to zero without loss of generality. We may now plot equation (28) for various values of x_A, the case of $x_A = 0$ corresponding to a straight line of constant gradient 1. The worldlines in both co-ordinate systems are illustrated in Figure 1 by the solid blue line.

For the second part of this question let us term the second particle as particle B. For particle B one has $\dot{x}_B = k$ (i.e. $x_B = kt$), where $k < 1$. The particle is now moving with constant velocity k and can be represented as a worldline of gradient $k < 1$ in the (t, x) co-ordinate system. In the (u, v) co-ordinate system one may write

$$u_B = t - kt = t(1 - k) \; , \tag{29}$$
$$v_B = t + kt = t(1 + k) \; , \tag{30}$$

from which one may conclude

$$\frac{u_B}{v_B} = \frac{1 - k}{1 + k}. \tag{31}$$

The condition that $(1 - k)/(1 + k) > 0$ implies that $|k| < 1$. Considering values of k in this range, the following condition on the gradient of the worldline may be obtained

$$\frac{u_B}{v_B} = \frac{1 - k}{1 + k} \begin{cases} < 1 \text{ if } k > 0 \text{ (Case B)} \; , \\ > 1 \text{ if } k < 0 \text{ (Case B}'\text{)} \; . \end{cases}$$

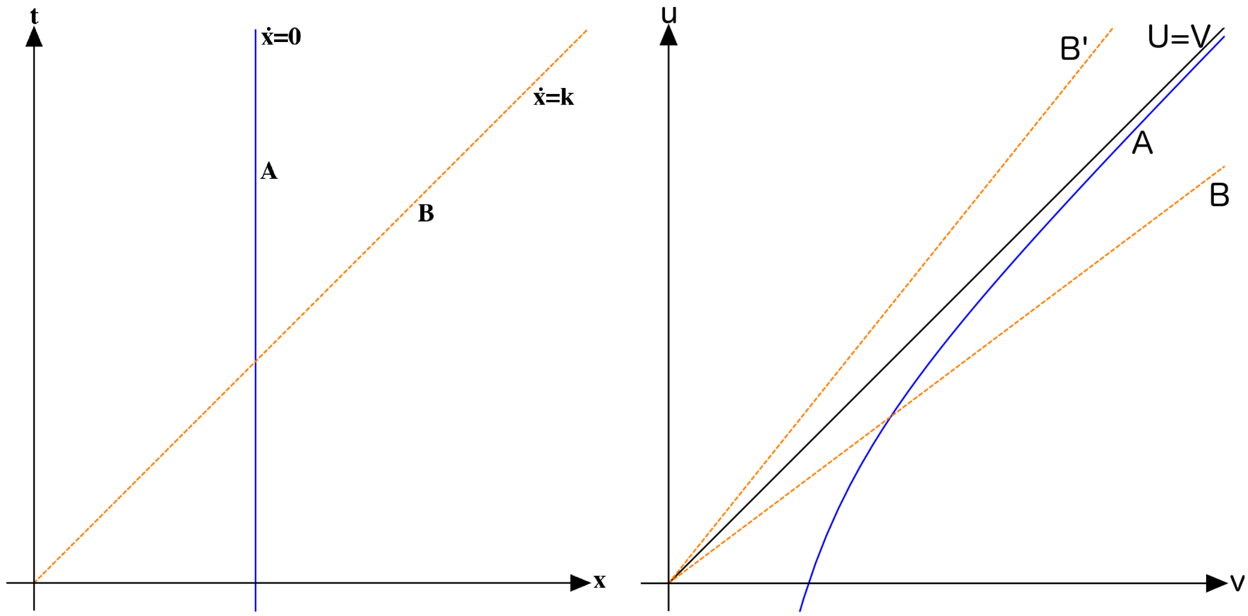

Figure 1: Worldliness for particles A and B in the (t, x) co-ordinate system (left) and the (u, v) co-ordinate system (right).

The worldlines in both co-ordinate systems are illustrated in Figure 1 by the dashed orange line.

For the final part of the question, for particle A, in the (t, x) co-ordinate system it is at rest. However, in the (u, v) co-ordinate system it is moving with constant velocity. For particle B, consider the limit $k \to 1$, whereby $\partial x/\partial t = 1$ and $\partial u_B/\partial v_B = 0$. In the (x, y) co-ordinate system the particle is moving with constant velocity, but in the limit $k \to 1$, in the (u, v) co-ordinate system this implies that the particle appears stationary (or the (v, u) co-ordinate system depending on how one labels the axes).