General Relativity: Solutions to exercises in Lecture XV

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Exercise 1

The simplest solution to the linearised Einstein equations is a plane wave of the form:

$$\bar{h}_{\mu\nu} = \Re\left\{A_{\mu\nu}\mathrm{e}^{i\kappa_\alpha x^\alpha}\right\} \ , \tag{1}$$

where $\Re$ denotes the real part, $\mathbf{A}$ is the "amplitude" tensor and κ is a null four-vector which satisfies $\kappa_\alpha \kappa^\alpha = 0$. In such a solution, the plane wave donated by equation (1) travels in the spatial direction $\vec{k} = (\kappa_x, \kappa_y, \kappa_z)/\kappa^0$, with frequency $\omega \equiv \kappa^0 = \left(\kappa_j \kappa^j\right)^{1/2}$. Determine the conditions such that the amplitude tensor $\mathbf{A}$ has only two linearly independent components, corresponding to the two states of polarisation of the gravitational waves.

Solution 1

$A_{\mu\nu}$ has at most 10 independent components. The solution to the linearised field equations $\Box\,\bar{h}_{\mu\nu} = 0$ is given by equation (1). Inserting this solution back into the linearised field equations yields:

$$\begin{aligned}
\Box\,\bar{h}_{\mu\nu} &= \eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{h}_{\mu\nu} \\
&= \eta^{\alpha\beta}\partial_\alpha\left(i\kappa_\beta\bar{h}_{\mu\nu}\right) \\
&= -\eta^{\alpha\beta}\kappa_\alpha\kappa_\beta\bar{h}_{\mu\nu} \\
&= -\left(\kappa_\alpha\kappa^\alpha\right)\bar{h}_{\mu\nu} \ = \ 0 \ ,
\end{aligned} \tag{2}$$

and thus we obtain the condition $\kappa_\alpha\kappa^\alpha = 0$. Defining $\kappa^\alpha = (\omega, \kappa^0\vec{k})$ and $\kappa_\alpha = (-\omega, \kappa^0\vec{k})$ we then immediately obtain $\omega \equiv \kappa^0 = \left(\kappa_j\kappa^j\right)^{1/2}$. Next, imposing the Lorentz (also know as de Donder) gauge $\bar{h}^{\mu\nu}{}_{,\mu} = 0$ we obtain:

$$\begin{aligned}
\partial_\mu\left(A^{\mu\nu}\mathrm{e}^{i\kappa_\alpha x^\alpha}\right) &= iA^{\mu\nu}\kappa_\mu\mathrm{e}^{i\kappa_\alpha x^\alpha} \\
&= i\left(\kappa_\mu A^{\mu\nu}\right)\mathrm{e}^{i\kappa_\alpha x^\alpha} \ = \ 0 \ ,
\end{aligned} \tag{3}$$

from which we obtain the condition:

$$\kappa_\mu A^{\mu\nu} = 0 \ , \tag{4}$$

i.e. the wave vector is orthogonal to $A^{\mu\nu}$. This condition constitutes a set of 4 algebraic equations and thus the number of degrees of freedom of $A_{\mu\nu}$ are reduced from 10 to 6. So far, whilst we have imposed the Lorentz gauge, we still have some remaining co-ordinate freedom. If we consider a co-ordinate transform of the form:

$$x'^{\mu} = x^{\mu} + \xi^{\mu} \ , \tag{5}$$

then $\Box\, x'^{\mu} = 0$ if $\Box\, \xi^{\mu} = 0$, which is also a wave equation. This has the solution

$$\xi_{\mu} = B_{\mu} \mathrm{e}^{i\kappa_{\alpha} x^{\alpha}} \ , \tag{6}$$

where κ_{α} is the wave vector and B_{μ} are constant coefficients. The remaining co-ordinate freedom allows us to transform from $A_{\mu\nu} \to A'_{\mu\nu}$ such that:

- ${A'}_{0\nu} = 0$ (wave amplitude transverse to its propagation direction)
- ${A'}^{\mu}{}_{\mu} = 0$ (traceless)

The choice of gauge sets $u^{\mu} {A'}_{\mu\nu} = 0$ for constant and timelike u^{μ}. The choice of Lorentz frame fixes u^{μ} to point along the time axis. Starting from the metric perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}\left([h_{\mu\nu}]^2\right) \ , \tag{7}$$

in the new co-ordinate system we obtain:

$$g'_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \ , \tag{8}$$

from which we immediately relate the metric perturbations between the two co-ordinate systems as:

$$h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \ . \tag{9}$$

Contracting both sides of this equation with $\eta^{\mu\nu}$ gives:

$$\begin{aligned} h' &= h - \xi^{\mu}{}_{\mu} - \xi^{\nu}{}_{\nu} \\ &= h - 2\,\xi^{\alpha}{}_{\alpha} \ . \end{aligned} \tag{10}$$

Now consider the trace-reversed part of $h'_{\mu\nu}$, namely $\bar{h}'_{\mu\nu}$, which may be simplified as follows:

$$\begin{aligned} h'_{\mu\nu} &= h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' \\ &= h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} - \frac{1}{2}\eta_{\mu\nu}\left(h - 2\,\xi^{\alpha}{}_{\alpha}\right) \\ &= h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\,\xi^{\alpha}{}_{\alpha} \\ &= \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\,\xi^{\alpha}{}_{\alpha} \ . \end{aligned} \tag{11}$$

Substituting our solution $\bar{h}_{\mu\nu} = A_{\mu\nu}\mathrm{e}^{i\kappa_{\alpha}x^{\alpha}}$ for the field equations and our solution $\xi_{\mu} = B_{\mu}\mathrm{e}^{i\kappa_{\alpha}x^{\alpha}}$ for the transformation between frames into equation (11) yields:

$$A'_{\mu\nu} = A_{\mu\nu} - i\kappa_{\nu} B_{\mu} - i\kappa_{\mu} B_{\nu} + i\eta_{\mu\nu}\kappa_{\alpha} B^{\alpha} \ . \tag{12}$$

We may now use the above equation to determine the components of $\mathbf{B}$ in terms of $A_{\mu\nu}$. Imposing the traceless condition implies contracting equation (12) with $\eta^{\mu\nu}$, yielding:

$$\begin{aligned} {A'}^{\mu}{}_{\mu} &= A^{\mu}{}_{\mu} - i\kappa_{\nu} B^{\nu} - i\kappa_{\mu} B^{\mu} + 4i\kappa_{\alpha} B^{\alpha} \\ &= A^{\mu}{}_{\mu} + 2i\kappa_{\mu} B^{\mu} \ = \ 0 \ , \end{aligned} \tag{13}$$

which gives the condition:

$$\kappa_{\mu} B^{\mu} = \frac{i}{2} A^{\mu}{}_{\mu} \ . \tag{14}$$

We next impose the transverse condition. Let us consider the temporal and spatial parts separately.

- For $\nu = 0$ we obtain:

$$A'_{00} = A_{00} - 2i\kappa_0 B_0 - i\kappa_\alpha B^\alpha = 0 \ . \tag{15}$$

However, we have previously derived equation (14) which upon substitution and simplification gives the temporal component of $\mathbf{B}$ as:

$$B_0 = -\frac{i}{2\kappa_0}\left(A_{00} + \frac{1}{2}A^\alpha{}_\alpha\right) \ . \tag{16}$$

- For $\nu = j$ we obtain:

$$\begin{aligned} A'_{0j} &= A_{0j} - i\kappa_0 B_j - i\kappa_j B_0 \\ &= A_{0j} - i\kappa_0 B_j - \frac{\kappa_j}{2\kappa_0}\left(A_{00} + \frac{1}{2}A^\alpha{}_\alpha\right) \ . \end{aligned} \tag{17}$$

From this we can solve for B_j, which upon simplification yields:

$$B_j = \frac{i}{2\kappa_0^2}\left[-2\kappa_0 A_{0,j} + \kappa_j\left(A_{00} + \frac{1}{2}A^\alpha{}_\alpha\right)\right] \ . \tag{18}$$

We now have the four constant coefficients for B_μ. We know that equations (17) and (18) satisfy the transverse condition and therefore $A'_{\mu\nu} \leftrightarrow A_{\mu\nu}$. The traceless condition implies 1 condition on the number of independent components of the amplitude tensor. For $\nu = 0$ the transverse condition implies $\kappa_\mu A^{\mu\nu} = 0$ which is redundant as we have already considered this. As such A'_{0j} (from the transverse condition) yields 3 conditions for $A'_{\mu\nu}$ (and therefore $A_{\mu\nu}$). Thus we conclude that the TT gauge gives us a further 4 constraints and so the number of linearly independent components of $A_{\mu\nu}$ is reduced from 6 to 2. Therefore the gravitational wave only has 2 independent states of polarisation, as required.

As an example, consider a gravitational wave travelling in the positive z-direction, where $k^\mu = (\omega, 0, 0, \kappa^z) \equiv (\omega, 0, 0, \omega)$. For such a null vector the conditions $k^\mu A_{\mu\nu} = 0$ and $A_{0\nu} = 0$ imply that $A_{3\nu} = 0$ also. Thus the only non-zero components are A_{11}, A_{12}, A_{21} and A_{22}. However, using the traceless condition $A^\mu{}_\mu = 0$ we also obtain $A_{22} = -A_{11}$. Finally, we know that by symmetry $A_{12} = A_{21}$ and thus we may write the amplitude tensor for such a gravitational wave as:

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \tag{19}$$

which only possesses 2 linearly independent components. It is important to note the that in the TT gauge we have $\bar{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu}$.

Exercise 2

The gauge satisfying the requirement of the first exercise is also refereed to as the TT (or transverse-traceless) gauge. Compute the non-zero components of the Riemann tensor in this gauge.

Solution 2

Recall from Exercise 1 that we defined the transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}$. From the consideration of nearby geodesics/particles separated by an infinitesimal distance ξ^{μ} (where $||\xi^{\mu}|| \ll 1$) one may calculate the geodesic deviation equation:

$$\frac{D^2\xi^{\mu}}{D\tau^2} = R^{\mu}{}_{\nu\alpha\beta}u^{\nu}u^{\alpha}\xi^{\beta} \ . \tag{20}$$

Let us calculate the RHS of equation (20) to first order in $h_{\mu\nu}$. Assuming neighbouring geodesics/particles vary slowly, we may express the four-velocity as a unit vector in the time direction plus corrections of $\mathcal{O}(h_{\mu\nu})$ and higher. Since the Riemann tensor is already first order in $h_{\mu\nu}$, corrections to u^{ν} can be ignored and we may set $u^{\nu} = (1, 0, 0, 0)$. With this the non-zero components of the geodesic deviation equation are found as

$$\frac{D^2\xi^{\mu}}{D\tau^2} = R^{\mu}{}_{00\beta}\,\xi^{\beta} \ . \tag{21}$$

Since $R^{\mu}{}_{00\beta} \neq 0$ this implies that $R_{\mu 00\beta} \neq 0$ also. From the symmetries of the Riemann curvature tensor we obtain:

$$R_{\mu 00\beta} = R_{0\mu 0\beta} = -R_{\mu 0\beta 0} = -R_{0\mu 0\beta} \ , \tag{22}$$

which are the only non-zero components. Thus there is only one independent component to the Riemann curvature tensor. We may write the expression for the Riemann curvature tensor as:

$$R_{\mu 00\beta} = \frac{1}{2}\left(h_{\mu\beta,00} + h_{00,\mu\beta} - h_{\mu 0,0\beta} - h_{\beta 0,0\mu}\right)^{\mathrm{TT}} , \tag{23}$$

where the superscript TT denotes evaluation of that quantity in the TT gauge. However, we know that $h_{\mu 0} = 0$ in the TT gauge and so the last three terms in equation (23) vanish, yielding:

$$R_{\mu 00\beta} = \frac{1}{2}\bar{h}^{\mathrm{TT}}_{\mu\beta,00} \ . \tag{24}$$

From the plane wave solution given in equation (1) we obtain:

$$\begin{aligned} \bar{h}^{\mathrm{TT}}_{\mu\beta,00} &= -\kappa_0\kappa_0\bar{h}^{\mathrm{TT}}_{\mu\beta} \\ &= -\omega^2\bar{h}^{\mathrm{TT}}_{\mu\beta} \end{aligned} \tag{25}$$

and therefore:

$$R_{\mu 00\beta} = -\frac{1}{2}\omega^2\bar{h}^{\mathrm{TT}}_{\mu\beta} \ . \tag{26}$$

Finally, in the TT gauge, and given our solution for $\bar{h}^{\mathrm{TT}}_{\mu\beta,00}$, we may assume $\bar{h}^{\mathrm{TT}}_{\mu\beta} \propto \mathrm{e}^{-i\omega t}$ and therefore:

$$R_{\mu 00\beta} \sim -\frac{1}{2}\omega^2\mathrm{e}^{-i\omega t} \ . \tag{27}$$