General Relativity: Solutions to exercises in Lecture XI

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All of the following exercises are to be considered in a special-relativistic context and assuming Cartesian co-ordinates where necessary.

Exercise 1

Within Special Relativity, consider a four-vector $\mathbf{V}$ with components:

$$\mathbf{V} = \sqrt{3}\,\mathbf{e}_t + \sqrt{2}\,\mathbf{e}_x \ . \tag{1}$$

Determine if $\mathbf{V}$ is timelike, null or spacelike. Compute the angles between $\mathbf{V}$ and the unit vectors $\mathbf{e}_t$ and $\mathbf{e}_x$.

Solution 1

First let us consider the inner-producut of $\mathbf{V}$ with itself:

$$\begin{aligned} \mathbf{V}\cdot\mathbf{V} &= \left(\sqrt{3}\right)^2 \mathbf{e}_t\cdot\mathbf{e}_t + \left(\sqrt{2}\right)^2 \mathbf{e}_x\cdot\mathbf{e}_x + 2\sqrt{2}\sqrt{3}\,\mathbf{e}_t\cdot\mathbf{e}_x \\ &= -3+2+0 \\ &= -1 \ < \ 0 \ , \end{aligned} \tag{2}$$

therefore $\mathbf{V}$ is timelike. For the angles, first let us consider the t-component. From the inner-product we may calculate the angle between $\mathbf{V}$ and $\mathbf{e}_t$ as:

$$\begin{aligned} \cos\theta_t &= \frac{\mathbf{V}\cdot\mathbf{e}_t}{|\mathbf{V}\cdot\mathbf{V}|^{1/2}|\mathbf{e}_t\cdot\mathbf{e}_t|^{1/2}} \\ &= -\sqrt{3} \ < -1 \ , \end{aligned} \tag{3}$$

which is not satisfied for any real θ_t. Similarly, for θ_x we find:

$$\cos\theta_x = \sqrt{2} \ > \ 1 \ , \tag{4}$$

which is also not satisfied for any real θ_x.

Exercise 2

A particle with rest mass m and four-momentum $\mathbf{p} = m\mathbf{v}$ is analysed by an observer with four-velocity $\mathbf{u}$. Compute the following:

- The total energy of the particle
- The kinetic energy of the particle
- The magnitude of the spatial momentum $p := \sqrt{p_i p^i}$
- The magnitude of the three-velocity $v := \sqrt{v_i v^i}$

Solution 2

Let us work in the rest frame of the observer. In this frame:

$$\begin{aligned} u^\alpha &= (1, \underline{0}) \ , && (5) \\ u_\alpha &= (-1, \underline{0}) \ , && (6) \\ p^\alpha &= (E, \underline{p}) \ , && (7) \\ p_\alpha &= (-E, \underline{p}) \ , && (8) \end{aligned}$$

where $\underline{p}$ is the three-momentum.

- The total energy may be obtained directly as

$$\begin{aligned} E &= -p_0 u^0 \\ &= -p_\alpha u^\alpha \ . && (9) \end{aligned}$$

- Starting form the expression for the total energy of a particle as $E^2 = p^2c^2 + m^2c^4$ one obtains the rest mass of the particle as:

$$\begin{aligned} m^2 &= E^2 - |\underline{p}|^2 \\ &= -p_0 p^0 - p_i p^i \\ &= -p_\alpha p^\alpha \ , && (10) \end{aligned}$$

  from which the kinetic energy of the particle may be directly derived as:

$$\begin{aligned} \text{K.E.} &= \frac{1}{2} m |\underline{v}|^2 \\ &= \frac{1}{2} \sqrt{-p_\alpha p^\alpha}\, v^2 \\ &= \frac{1}{2} \sqrt{-p_\alpha p^\alpha} \sqrt{-v_i v^i} \ . && (11) \end{aligned}$$

- Starting again from the expression for the total energy of the particle, the magnitude of the three-momentum may be calculated as:

$$\begin{aligned} p &= \left(E^2 - m^2\right)^{1/2} \\ &= \left[(p_\alpha u^\alpha)^2 + p_\beta p^\beta \right]^{1/2} \ . && (12) \end{aligned}$$

- Finally, the magnitude of the three-velocity may be calculated directly as:

$$\begin{aligned} v &= \frac{p}{E} \\ &= \frac{1}{E}\left(E^2 - m^2\right)^{1/2} \\ &= \left(1 - \frac{m^2}{E^2}\right)^{1/2} \\ &= \left[1 + \frac{p_\alpha p^\alpha}{\left(p_\beta u^\beta\right)^2}\right]^{1/2} \quad . \end{aligned} \tag{13}$$

Exercise 3

Define the four-acceleration of a particle with four-velocity $\mathbf{u}$ as

$$a^\mu := \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} \ , \tag{14}$$

where τ is the proper time. Show that $\mathbf{a} \cdot \mathbf{u} = 0$, i.e. that the acceleration is orthogonal to the four-velocity. What does this mean in a frame co-moving with the particle?

Solution 3

Let us start with the following identity for a particle in General Relativity:

$$u_\mu u^\mu = -1 \ , \tag{15}$$

from which it immediately follows that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(u_\mu u^\mu\right) = 0 \ . \tag{16}$$

Expanding the above expression yields:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(u_\mu u^\mu\right) &= 2\,\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau}\,u^\mu \\ &= 2\,a_\mu u^\mu \\ &= 0 \ . \end{aligned} \tag{17}$$

We may thus conclude that $\mathbf{a} \cdot \mathbf{u} = 0$, as required.

In a frame co-moving with the particle, $u^\mu = (1, \underline{0})$. In this frame $a_\mu u^\mu = 0$ implies that $a_0 = 0$ by necessity, but that the spatial components of the four-acceleration, a_i, are arbitrary.