General Relativity: Solutions to exercises in Lecture X

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Exercise 1

Show that the second covariant derivatives of a scalar field commute, i.e. that

$$\nabla_\alpha \nabla_\beta \phi = \nabla_\beta \nabla_\alpha \phi \ . \tag{1}$$

Obtain the expressions for the following third derivatives: $\nabla_\alpha \nabla_{(\beta} \nabla_{\gamma)} \phi$ and $\nabla_{[\alpha} \nabla_{\beta]} \nabla_\gamma \phi$. [Hint: remember that the covariant derivative of a scalar field is a vector.]

Solution 1

- For the first part of the question we consider the action of the covariant derivatives in order, remembering that the covariant derivative of a scalar is simply the partial derivative acting on that scalar. This yields:

$$\begin{aligned} \nabla_\alpha \nabla_\beta \phi &= \left(\phi_{;\beta}\right)_{;\alpha} \\ &= \left(\phi_{,\beta}\right)_{;\alpha} \\ &= \phi_{,\alpha\beta} - \phi_{,\delta} \Gamma^{\delta}{}_{\alpha\beta} \ . \end{aligned} \tag{2}$$

  Since the Christoffel symbols are symmetric in their lower indices (torsion-free) and partial derivatives commute, we may conclude that $\nabla_\alpha \nabla_\beta \phi = \nabla_\beta \nabla_\alpha \phi$, as required.

- For the second part of the question, let us first define the covariant vector $W_\gamma \equiv \nabla_\gamma \phi$. Now consider

$$\nabla_\alpha \nabla_\beta \nabla_\gamma \phi = \nabla_\alpha \nabla_\beta W_\gamma \ , \tag{3}$$

  and similarly

$$\nabla_\alpha \nabla_\gamma \nabla_\beta \phi = \nabla_\alpha \nabla_\gamma W_\beta \ . \tag{4}$$

  Using the result of the first part of the question we may write

$$\nabla_\beta W_\gamma = \nabla_\gamma W_\beta \ . \tag{5}$$

  Employing the above we may now write

$$\begin{aligned} \nabla_\alpha \nabla_{(\beta} \nabla_{\gamma)} \phi &= \frac{1}{2} \nabla_\alpha \left(\nabla_\beta W_\gamma + \nabla_\gamma W_\beta \right) \\ &= \nabla_\alpha \nabla_\beta W_\gamma \\ &= \nabla_\alpha \nabla_\beta \nabla_\gamma \phi \ . \end{aligned} \tag{6}$$

- For the final part of the question let us directly expand the expression in question:

$$\begin{aligned}\nabla_{[\alpha}\nabla_{\beta]}\nabla_\gamma\phi &= \nabla_{[\alpha}\nabla_{\beta]}W_\gamma \\ &= \frac{1}{2}\left(\nabla_\alpha\nabla_\beta W_\gamma - \nabla_\beta\nabla_\alpha W_\gamma\right) \\ &= \frac{1}{2}R^\delta{}_{\alpha\beta\gamma}W_\delta \\ &= \frac{1}{2}R^\delta{}_{\alpha\beta\gamma}\phi_{;\gamma} \\ &= \frac{1}{2}R^\delta{}_{\alpha\beta\gamma}\phi_{,\gamma}\ .\end{aligned} \tag{7}$$

Exercise 2

Prove that for any second-rank tensor, the covariant derivative commutes, i.e. that

$$\nabla_\alpha\nabla_\beta V^{\alpha\beta} = \nabla_\beta\nabla_\alpha V^{\alpha\beta}\ . \tag{8}$$

Solution 2

First recall the fact that $\nabla_\beta V^{\alpha\beta}$ is a rank-1 contravariant tensor, thus we may define $W^\mu \equiv \nabla_\beta V^{\mu\beta}$. Next, let us write explicitly the expression for W^μ as follows:

$$W^\mu = \partial_\beta V^{\mu\beta} + \Gamma^\mu{}_{\beta\delta}V^{\delta\beta} + \Gamma^\beta{}_{\beta\delta}V^{\mu\delta}\ . \tag{9}$$

We may now write the covariant derivatives acting on $V^{\alpha\beta}$ as:

$$\begin{aligned}\nabla_\alpha\nabla_\beta V^{\alpha\beta} &= \nabla_\alpha W^\alpha \\ &= \partial_\alpha W^\alpha + \Gamma^\alpha{}_{\alpha\gamma}W^\gamma\ .\end{aligned} \tag{10}$$

Thus we may write the LHS and RHS of equation (8) as:

$$\begin{aligned}\nabla_\alpha\nabla_\beta V^{\alpha\beta} = \partial_\alpha\partial_\beta V^{\alpha\beta} &+ \partial_\alpha\left(\Gamma^\alpha{}_{\beta\delta}V^{\delta\beta}\right) + \partial_\alpha\left(\Gamma^\beta{}_{\beta\delta}V^{\alpha\delta}\right) + \Gamma^\alpha{}_{\alpha\gamma}\partial_\beta V^{\gamma\beta} + \Gamma^\alpha{}_{\alpha\gamma}\Gamma^\gamma{}_{\beta\delta}V^{\delta\beta} \\ &+ \Gamma^\alpha{}_{\alpha\gamma}\Gamma^\beta{}_{\beta\delta}V^{\gamma\delta}\ ,\end{aligned} \tag{11}$$

and

$$\begin{aligned}\nabla_\beta\nabla_\alpha V^{\alpha\beta} = \partial_\beta\partial_\alpha V^{\alpha\beta} &+ \partial_\beta\left(\Gamma^\alpha{}_{\alpha\delta}V^{\delta\beta}\right) + \partial_\beta\left(\Gamma^\beta{}_{\alpha\delta}V^{\alpha\delta}\right) + \Gamma^\beta{}_{\beta\gamma}\partial_\alpha V^{\alpha\gamma} + \Gamma^\beta{}_{\beta\gamma}\Gamma^\alpha{}_{\alpha\delta}V^{\delta\gamma} \\ &+ \Gamma^\beta{}_{\beta\gamma}\Gamma^\gamma{}_{\alpha\delta}V^{\alpha\delta}\ .\end{aligned} \tag{12}$$

Consider each term in eqns. (11) & (12), and let us refer to these equations as L and R respectively, along with the indices 1–6 indicating terms 1–6 respectively in each expression. Under $\alpha \leftrightarrow \beta$:

$$\begin{aligned}L_1 &= R_1\ , \\ L_2 &= R_3\ , \\ L_3 &= R_2\ , \\ L_4 &= R_4\ , \\ L_5 &= R_6\ , \\ L_6 &= R_5\ .\end{aligned}$$

We may thus conclude that $\nabla_\alpha\nabla_\beta V^{\alpha\beta} = \nabla_\beta\nabla_\alpha V^{\alpha\beta}$, as required.

Exercise 3

Optional: Find the matrix of the Lorentz transformations corresponding to a boost v^x in the x-direction followed by a boost v^y in the y-direction. What happens if the order of the boosts is reversed?

Solution 3

Let us write the Lorentz boost matrices in the x- and y-directions respectively as:

$$\Lambda_x = \begin{pmatrix} \gamma_x & \gamma_x v_x & 0 & 0 \\ \gamma_x v_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{13}$$

and

$$\Lambda_y = \begin{pmatrix} \gamma_y & 0 & \gamma_y v_y & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_y v_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

For the first combination of boosts we obtain:

$$\Lambda_y \Lambda_x = \begin{pmatrix} \gamma_x \gamma_y & \gamma_x \gamma_y v_x & \gamma_y v_y & 0 \\ \gamma_x v_x & \gamma_x & 0 & 0 \\ \gamma_x \gamma_y v_y & \gamma_x \gamma_y v_x v_y & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Similarly, for the reverse transformation we find:

$$\Lambda_x \Lambda_y = \begin{pmatrix} \gamma_x \gamma_y & \gamma_x v_x & \gamma_x \gamma_y v_y & 0 \\ \gamma_x \gamma_y v_x & \gamma_x & \gamma_x \gamma_y v_x v_y & 0 \\ \gamma_y v_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Clearly $\Lambda_y \Lambda_x \neq \Lambda_x \Lambda_y$ and so the transformations do not commute.